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IDENTIFICATION IN INSTRUMENTAL VARIABLES MODELS: THE CENTRAL ROLE OF ABADIE'S KAPPA

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We study instrumental variables models characterized by: (i) Unobserved heterogeneity consisting of potential outcomes and response types that describe how the instrument determines treatment choice; (ii) Conditional independence of the instrument and the unobserved heterogeneity; and (iii) Convex restrictions on the distribution of unobserved heterogeneity. We show certain causal parameters are identified in these models if and only if a version of the kappa of Abadie (2003) exists. Our identification results are constructive in yielding estimating moment conditions. Focusing on a leading special case, we develop asymptotically normal estimators based on a doubly robust version of these moment conditions.

KEYWORDS: Potential outcomes, instrumental variables, identification.

1. INTRODUCTION

POTENTIAL OUTCOMES MODELS are the leading framework for identifying and estimating causal effects with heterogeneous treatment responses. Originally developed for randomized experiments by Neyman (1990), they were shown by Rubin (1974) to provide a foundation for causal inference in observational studies. Their central role in instrumental variables models was established by Imbens and Angrist (1994) who showed the importance of restricting the manner in which an instrument determines treatment choice. A subsequent literature has built on this work by developing identifying restrictions for a wide range of empirically relevant settings, including applications with ordered and unordered treatments, multiple instruments, and mediating variables.

In this paper, we propose a class of instrumental variables models that unifies and expands upon these identification strategies. The core assumptions of our framework are: (i) Unobserved heterogeneity consists of potential outcomes and response types that describe how the instrument determines treatment choice; (ii) Instrumental variables are conditionally independent of unobserved heterogeneity in both potential outcomes and response types; and (iii) The distribution of unobserved heterogeneity belongs to a convex set. Models within the scope of this framework include those that secure identification by imposing assumptions that restrict the possible response types or, in the terminology of

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Frangakis and Rubin (2002), principal strata. These encompass, among many others, the monotonicity restriction of Angrist and Imbens (1995), the partial monotonicity requirement of Mogstad, Torgovitsky, and Walters (2021), and the revealed preference restrictions of Kline and Walters (2016) and Pinto (2021). Additional examples of identifying assumptions that fall within our framework include the sequential ignorability requirement of Imai, Keele, and Yamamoto (2010) and the comparative compliers assumption of Mountjoy (2022).

Our main contribution is to show that these models share a common necessary and sufficient condition for identifying causal parameters that can be expressed as expectations of functions of unobserved heterogeneity and covariates. Intuitively, our results show that identification holds if and only if a suitable generalization of the kappa of Abadie (2003) exists. More formally, we show that identification is equivalent to the function whose expectation we seek belonging to the closure of the range of a linear operator Υ that maps functions of observables to functions of unobservables. Identification is therefore tantamount to the existence of a sequence of functions $\{\kappa_j\}$ of observable variables such that $\Upsilon(\kappa_j)$ suitably approximates the function of unobserved heterogeneity whose expectation we wish to identify. Critically, the map Υ is the same across all the models in our framework, but the sense in which $\Upsilon(\kappa_j)$ must provide a suitable approximation depends on the restrictions imposed by the specific model. The functions $\{\kappa_j\}$ may be viewed as a generalization of Abadie's kappa, and our results can therefore be interpreted as showing that the identification strategy of Abadie (2003) is central to identifying causal parameters in a class of instrumental variable models that extends beyond Imbens and Angrist (1994).

Our identification result is constructive. We show that if the parameter of interest is identified, then it equals the limit of the expectations of the approximating sequence $\{\kappa_j\}$. This characterization suggests a simple estimation strategy: Estimate the relevant $\{\kappa_j\}$ and compute their sample averages. Establishing asymptotic normality when the nuisance parameters $\{\kappa_j\}$ are estimated using machine learning methods, however, often requires characterizing orthogonal scores that depend on the specific restrictions being imposed by the model. For estimation, we therefore focus on a leading special case of our framework in which the identifying restrictions do not depend on the distribution of observables. In this context, we derive a double robust moment condition and follow Smucler, Rotnitzky, and Robins (2019), CEIN+ (2022), and Chernozhukov, Newey, and Singh (2022) by employing ℓ_1-regularization to estimate nuisance parameters. We show that the resulting estimators are asymptotically normally distributed and that bootstrap based inference is asymptotically valid. We provide code for assessing whether a parameter is identified and implementing our estimator when it is for the special case of applications with discrete instruments and restrictions that only limit response types.[1]

When specialized to existing models within our framework, our results both recover existing results and generate new insights. For example, in the setting of Imbens and Angrist (1994), our identification results recover Abadie (2003), while our estimation results yield the double robust moment condition used by Tan (2006) and Singh and Sun (2022). In the related discrete instrument models studied by Heckman and Pinto (2018), our results provide conditions under which an assumption they show to be sufficient for identification is also necessary.[2] In other applications, our results can characterize what parameters are

[1]Found at the following public GitHub repository: https://github.com/mnavjeev/IdentificationCheck.git.

[2]Corollary C-1 in Heckman and Pinto (2018) provides necessary and sufficient conditions for linear restrictions implied by the model to deliver identification. Our results in contrast provide necessary and sufficient conditions for identification that reflect all the restrictions of the model.

identified by existing assumptions. Mogstad, Torgovitsky, and Walters (2021), for example, introduce a partial monotonicity assumption in order to interpret the two stage least squares estimand causally. As an illustration, we show how our results extend their analysis by readily characterizing all other causal estimands that are identified by the same partial monotonicity requirement.

Our results are not only useful in the context of existing models. By providing a transparent connection between assumptions and the parameters they identify, our results are also helpful for developing, identifying, and estimating new causal models. We illustrate this point with an empirical reanalysis of the Moving to Opportunity (MTO) experiment (Orr, Feins, Jacob, and Beecroft (2003), Kling, Liebman, and Katz (2007), Orr (2011)). Specifically, we evaluate a conjecture by LLKD+ (2008) who suggested that improved mental health may play an important role in the causal mechanism through which moving from high to low-poverty neighborhoods impacts economic outcomes. Guided by our necessary and sufficient conditions for identification, we devise a model that enables us to identify and estimate the mediating effects of improved mental health for different subpopulations. The resulting model is new and distinct from the mediation analysis in Frölich and Huber (2017), Ding and Lu (2017), and Rudolph, Williams, and Díaz (2024).

This paper contributes to the literature on potential outcomes models. Our analysis appears to be the first to establish the unifying role of a common necessary and sufficient condition for identification. In independent and subsequent work, Goff (2025) studies necessary and sufficient conditions for identifying treatment effects conditional on response type in the discrete instrument model of Heckman and Pinto (2018). Angrist, Santos, and Tecchio (2026) builds on our results by introducing new causal parameters and deriving transparent necessary and sufficient conditions for their identification in applications with a single Bernoulli instrument. Our analysis is further related to work providing analytical (Manski (1990)) or computational (Balke and Pearl (1997), Mogstad, Santos, and Torgovitsky (2018)) bounds for partially identified parameters. By showing that point identification turns on a single linear equation, our results provide a simple way to determine when such bounds collapse to a point.

The rest of the paper is organized as follows. Section 2 introduces the class of models we study, and Section 3 illustrates our results in the context of the MTO experiment. Our main theoretical results are in Sections 4 and 5, which discuss identification and estimation.

2. THE MODEL

We consider applications with a scalar outcome $Y \in \mathbf{Y} \subseteq \mathbf{R}$, a discrete treatment $T \in \mathbf{T} \equiv \{t_1, \ldots, t_d\}$, an instrument $Z \in \mathbf{Z}$, and covariates $X \in \mathbf{X}$. The instrument Z and covariates X may be vectors (e.g., $\mathbf{Z}, \mathbf{X} \subseteq \mathbf{R}^k$) or, as we discuss in the Appendix, have a more general structure. We model unobserved heterogeneity through the potential outcomes $Y^* \equiv (Y^*(t_1), \ldots, Y^*(t_d))$ and a response type $T^* : \mathbf{Z} \to \{t_1, \ldots, t_d\}$ that describes how the instrument determines a unit's treatment decision. The observed treatment and outcome are given by $T = T^*(Z)$ and $Y = Y^*(T)$.

In order to state the assumptions that characterize our model, we first define

$$L^p(Q) \equiv \{f : \|f\|_{Q,p} < \infty\}, \qquad \|f\|_{Q,p} \equiv \left(\int |f|^p \, dQ\right)^{\frac{1}{p}};$$

that is, $L^p(Q)$ denotes the set of functions that have a finite pth moment under Q. We say that a distribution Q is absolutely continuous with respect to (w.r.t.) a distribution Q', denoted $Q \ll Q'$, if Q assigns zero probability to any event to which Q' assigns zero probability. Whenever $Q \ll Q'$, Q has a density w.r.t. Q' that we denote by dQ/dQ'. We also let Q_0 denote the distribution of (Y^*, T^*, Z, X) and P the distribution of (Y, T, Z, X).

Given the introduced notation, we impose the following two assumptions.

ASSUMPTION 2.1: *(i) $(Y^*, T^*, Z, X) \sim Q_0$ with $Y^* \equiv (Y^*(t_1), \dots, Y^*(t_d)) \in \mathbf{Y}^* \subseteq \mathbf{R}^d$, $Z \in \mathbf{Z}$, $X \in \mathbf{X}$, and $T^* \in \mathbf{T}^*$ with $\mathbf{T}^*$ a set of functions from $\mathbf{Z}$ to $\mathbf{T} \equiv \{t_1, \dots, t_d\}$; (ii) We observe $T = T^*(Z)$, $Y = Y^*(T)$, Z, and X with $(Y, T, Z, X) \sim P$.*

ASSUMPTION 2.2: *(i) $(Y^*, T^*) \perp\!\!\!\perp Z|X$ under Q_0; (ii) $Q_0 \ll \mu$ for some separable probability measure μ; (iii) $dQ_0/d\mu$ belongs to a set $\mathcal{Q} \subseteq L^1(\mu)$ for $\mathcal{Q}$ a closed convex subset of Banach space $(\mathbf{Q}, \|\cdot\|_{\mathbf{Q}})$ with $\|\cdot\|_{\mathbf{Q}}$ (weakly) stronger than $\|\cdot\|_{\mu,1}$.*

Assumption 2.1 formalizes our setup. Assumption 2.2(i) requires that Z be exogenous, while Assumption 2.2(ii) imposes that the distribution of (Y^*, T^*, Z, X) be absolutely continuous with respect to a measure μ. A choice of μ corresponds to a choice of support restrictions imposed by the model. For example, since Q_0 must assign zero probability to any event to which μ assigns zero probability, we may employ μ to rule out certain realizations of T^*, such as defiers in Imbens and Angrist (1994). Assumption 2.2(iii) accommodates additional convex restrictions. These restrictions may include regularity conditions that ensure the parameter of interest is well-defined (see Example 2.1) and more substantive identifying assumptions (see Section 3). We note that μ, $\mathcal{Q}$, and $\mathbf{Q}$ may all depend on P, though do not make the dependence explicit, that is, they may be identified but not known. We also highlight that different measures μ can yield the same model by, for example, imposing the same support restrictions through Assumption 2.2(ii). This flexibility is often helpful, as it allows us to select measures μ that simplify calculations; see Remark 4.1.

The distribution Q_0 of (Y^*, T^*, Z, X) induces, through Assumption 2.1, the distribution P of (Y, T, Z, X). Absent restrictive assumptions, Q_0 is not identified because there are alternative distributions Q for (Y^*, T^*, Z, X) that induce P. We refer to any such distribution Q as observationally equivalent to Q_0. The identified set for Q_0 is defined as

$$\Theta_0 \equiv \left\{Q : Q \text{ is obs. equiv. to } Q_0, \left(Y^*, T^*\right) \perp\!\!\!\perp Z|X \text{ under } Q, Q \ll \mu, \frac{dQ}{d\mu} \in \mathcal{Q}\right\};$$

that is, Θ_0 is the set of distributions for (Y^*, T^*, Z, X) that induce P and satisfy the requirements imposed on Q_0 in Assumption 2.2.

Our primary goal is to study the identification and estimation of features of the true distribution Q_0. Concretely, for any given function ℓ of (Y^*, T^*, X) we let

$$\lambda_Q \equiv E_Q\left[\ell\left(Y^*, T^*, X\right)\right], \tag{1}$$

where the Q subscript emphasizes that the expectation is taken with respect to a distribution Q that may not equal Q_0. We study identification of the expectation of $\ell(Y^*, T^*, X)$ under Q_0, which we denote by λ_{Q_0}. This expectation is identified if and only if λ_Q takes the same value for every $Q \in \Theta_0$. We emphasize that ℓ can depend on P, though we leave such dependence implicit, that is, ℓ must be an identified function but not necessarily known.

2.1. *Examples*

In order to fix ideas, we introduce examples that highlight the flexibility of our setup.

EXAMPLE 2.1: Following Rosenbaum and Rubin (1983), suppose treatment $T \in \{0, 1\}$ is Bernoulli and $Y^* \equiv (Y^*(0), Y^*(1))$ is independent of T conditional on X. To map this setting into our framework, let $Z = T$, select μ to satisfy $\mu(T^*(Z) = Z) = 1$, and note Assumption 2.2(i) is then equivalent to $Y^* \perp\!\!\!\perp T|X$. The identified set Θ_0 consists of the distributions Q for (Y^*, T, X) that induce the distribution P of observables and satisfy the maintained assumptions, while Q_0 denotes the true distribution of (Y^*, T, X). Setting $\ell(Y^*, T^*, X) = Y^*(1) - Y^*(0)$, we can write the average treatment effect (ATE) as the expectation of ℓ under Q_0. In this model, the ATE is identified, which is equivalent to $\lambda_Q \equiv E_Q[Y^*(1) - Y^*(0)]$ being constant in $Q \in \Theta_0$. Ensuring the ATE is well-defined requires us to assume $(Y^*(0), Y^*(1))$ has a first moment, which can be imposed through Assumption 2.2(iii). Other choices of ℓ, such as $\ell(Y^*, T^*, X) = 1\{Y^*(1) \geq Y^*(0)\}$, lead to parameters λ_{Q_0} that are not identified because λ_Q changes with $Q \in \Theta_0$.

EXAMPLE 2.2: Consider a special case of Imbens and Angrist (1994) in which $T \in \{0, 1\}$ and $Z \in \{0, 1\}$ are Bernoulli. In this context, T^* maps $\mathbf{Z} \equiv \{0, 1\}$ to $\mathbf{T} \equiv \{0, 1\}$. Assumption 2.2(ii) allows us to impose that the instrument weakly induce individuals into treatment by setting μ to satisfy $\mu(T^*(1) \geq T^*(0)) = 1$. If we let $\ell(Y^*, T^*, X) = 1\{T^*(1) > T^*(0)\}$, then λ_Q equals the probability of compliers under any distribution Q for (Y^*, T^*, Z, X), while λ_{Q_0} equals the true probability of compliers. The probability of compliers is identified because λ_Q is constant in $Q \in \Theta_0$. To study parameters such as the marginal distribution of Y^* for compliers, as in Imbens and Rubin (1997), we write

$$E_{Q_0}\big[f\big(Y^*(t)\big)|T^*(1) > T^*(0)\big] = \frac{E_{Q_0}\big[f\big(Y^*(t)\big)1\big\{T^*(1) > T^*(0)\big\}\big]}{E_{Q_0}\big[1\big\{T^*(1) > T^*(0)\big\}\big]} \tag{2}$$

for any given function f and $t \in \{0, 1\}$. The conditional expectation in (2) equals the ratio of two parameters with the structure in (1) with ℓ known. Our results provide necessary and sufficient conditions for the identification of the numerator, the denominator, and also the ratio in (2); see Remark 4.3 for details.

EXAMPLE 2.3: Heckman and Vytlacil (2005) study a model in which treatment is determined by $T = 1\{f(X, Z) \geq \xi\}$ with f an unknown continuous function, ξ unobservable, and $(Y^*(0), Y^*(1), \xi) \perp\!\!\!\perp Z|X$. Assuming that Z is a scalar and $f(X, \cdot)$ is increasing, we may map this model into our framework by letting $T^* \equiv 1\{f(X, \cdot) \geq \xi\}$ and imposing[3]

$$\mu\Big(\lim_{z\downarrow z'} T^*(z) = T^*\big(z'\big) \text{ and } T^*(z) \geq T^*\big(z'\big) \text{ for all } z \geq z'\Big) = 1.$$

The literature has focused on causal parameters that can be expressed as expectations of $h(Y^*, \xi, X)$ for some function h. Because T^* may not be an invertible function of ξ, these expectations may not map into the parameters in (1) that we study. However, under regularity conditions, it is possible to show that a necessary condition for the expectation of

[3]The upper semicontinuity of T^* is due to the continuity of $f(X, \cdot)$. The case in which $f(X, \cdot)$ is not monotonic corresponds to imposing $T^*(z) \geq T^*(z')$ whenever $P(T = 1|Z = z, X) \geq P(T = 1|Z = z', X)$.

$h(Y^*, \xi, X)$ to be identified is that h depend on ξ only through T^*. Hence, our characterization of identification of (1) characterizes identification of causal parameters that can be expressed as expectations of $h(Y^*, \xi, X)$ for some function h. Similarly, our framework can accommodate related structural models, such as those in Lee and Salanié (2018).

EXAMPLE 2.4: Kline and Walters (2016) evaluate the cost effectiveness of the Head Start Program. In their analysis, $Z \in \{0, 1\}$ denotes whether an individual was offered to attend a Head Start school and T can take three values: Attend a Head Start school (h), attend other schools (c), or receive home care (n). Here, T^* maps $\mathbf{Z} \equiv \{0, 1\}$ to $\mathbf{T} \equiv \{h, c, n\}$ and we can therefore characterize T^* as a vector $T^* = (T^*(0), T^*(1))$ taking values in $\{h, c, n\} \times \{h, c, n\}$. The main identification assumption in Kline and Walters (2016) is that receiving an offer to attend a Head Start school may only induce individuals to attend a Head Start school. Formally, they require that $T^* = (T^*(0), T^*(1))$ belong to

$$\mathbf{R}^* \equiv \left\{ \begin{pmatrix} n \\ h \end{pmatrix}, \begin{pmatrix} c \\ h \end{pmatrix}, \begin{pmatrix} n \\ n \end{pmatrix}, \begin{pmatrix} c \\ c \end{pmatrix}, \begin{pmatrix} h \\ h \end{pmatrix} \right\},$$

which can be mapped into our framework by setting μ to satisfy $\mu(T^* \in \mathbf{R}^*) = 1$. More generally, in applications with a discrete instrument we may always impose support restrictions on T^* by demanding that $\mu(T^* \in \mathbf{R}^*) = 1$ for some finite set of vectors $\mathbf{R}^* \equiv \{t_1^*, \ldots, t_r^*\}$. Through this observation, our framework accommodates the unordered monotonicity condition of Heckman and Pinto (2018), the analysis of the MTO experiment by Pinto (2021), and the survey non-response model of DHLM+ (2021).

EXAMPLE 2.5: Mogstad, Torgovitsky, and Walters (2021) propose a partial monotonicity condition that yields a causal interpretation for two stage least squares (TSLS) in applications with multiple instruments. In an empirical reexamination of Carneiro, Heckman, and Vytlacil (2011), the authors consider a setting in which $T \in \{0, 1\}$ indicates whether an individual attended college, Y represents log average hourly wage, and $Z = (C, W)$ with $C \in \{0, 1\}$ indicating whether a college is present in the county of residence at age 14 and W denoting average log earnings in the county of residence at age 17. Mogstad, Torgovitsky, and Walters (2021) suppose that increasing C induces individuals into treatment, while increasing W induces individuals out of treatment. Their requirement may be mapped into our framework by selecting μ to satisfy

$$\mu\big(T^*(1, w) \geq T^*(0, w) \text{ and } T^*(c, w) \leq T^*\big(c, w'\big) \text{ for all } c \in \{0, 1\}, w \geq w'\big) = 1. \tag{3}$$

Restrictions on the response types that are analogous to (3) were also employed in the empirical study of the returns to 2-year colleges by Mountjoy (2022).

EXAMPLE 2.6: Mediation analysis aims to identify how a treatment affects an outcome through intermediate variables called mediators. Angrist, Autor, and Pallais (2022), for instance, hypothesize that early engagement with a 4-year college mediates the impact of student grants on graduation rates. Letting $D \in \{0, 1\}$ indicate whether a student is awarded a grant, $M \in \{e, ne\}$ whether she engaged (e) early with a 4-year college or not (ne), and $Y \in \{0, 1\}$ whether she graduated college, we may map their study into our framework by letting $T = (D, M)$ and $Z = D$. Potential outcomes Y^* are then indexed by $t = (d, m) \in \{0, 1\} \times \{e, ne\}$, while T^* is a function mapping $\mathbf{Z} \equiv \{0, 1\}$ to $\mathbf{T} \equiv \{0, 1\} \times \{e, ne\}$. Assumptions 2.2(i,ii) may then be employed to impose, for example, the sequential ignorability requirement of Imai, Keele, and Yamamoto (2010). Angrist, Santos, and Tecchio (2026) build on the mediation model we develop in Section 3 to assess the mediating role of early engagement discussed in Angrist, Autor, and Pallais (2022).

3. MOVING TO OPPORTUNITY

As a preview of our results, we first illustrate the ability of our methods to develop, identify, and estimate a causal model in the context of an empirical analysis of MTO. MTO was an experiment in which households living in high-poverty neighborhoods were offered vouchers that incentivized them to relocate to low-poverty neighborhoods. The intervention generated substantial and long-lasting improvements in mental health but more nuanced effects on economic outcomes (Katz, Kling, and Liebman (2001), LSGA+ (2012), Pinto (2021)).

While mental health and economic outcomes may be mutually causally related, Kling, Liebman, Katz, and Sanbonmatsu (2004) find reduced stress due to relocation is the primary cause of improved mental health, rather than improvements in economic outcomes. Building on these results, LLKD+ (2008) conjecture that improved mental health may be an important mediator through which the MTO intervention affects labor market participation. In what follows, we revisit the MTO experiment to examine this hypothesis.

We use data on mental health and labor market outcomes collected in the Interim Evaluation (2001–2002, 4–7 years after randomization) and control for baseline variables surveyed at the start of the intervention. We let $Z \in \{0, 1\}$ indicate whether a voucher was offered, Y denote an outcome of interest, and $D \in \{0, 1\}$ indicate whether the household relocated to a low-poverty neighborhood. The measure of mental health is an indicator, $M \in \{0, 1\}$, of whether the sampled adult reported feeling calm and peaceful all or most of the time during the previous thirty days. We set $T = (D, M)$ and let $Y^*(t)$ depend on $t = (d, m)$ to reflect that mental health and relocation can affect outcomes. Covariates, X, consist of experimental site indicators and household and neighborhood characteristics.

3.1. *Learning About Types*

We begin by studying features of the distribution of response types T^*. In particular, for functions ℓ of (T^*, X), we first identify and estimate parameters with the structure

$$E_{Q_0}\big[\ell\big(T^*, X\big)\big]. \tag{4}$$

By selecting μ in Assumption 2.2(ii) to restrict the support of T^*, our model enables us to restrict how a voucher offer affects relocation decisions and mental health. Under such support restrictions, our results imply that (4) is identified if and only if there exists a function ν of (T, Z, X) such that for all t^* in the support of T^* and all X we have

$$\sum_{t\in\mathbf{T}}\sum_{z\in\mathbf{Z}} 1\big\{t^*(z) = t\big\}\nu(t, z, X) = \ell\big(t^*, X\big); \tag{5}$$

see Corollary 4.3. Any solution ν to (5) can be used to identify (4) by constructing the function $\kappa(t, z, X) = \nu(t, z, X)/P(Z = z|X)$, which our results show satisfies

$$E_{Q_0}\big[\ell\big(T^*, X\big)\big] = E_P\big[\kappa(T, Z, X)\big]. \tag{6}$$

Guided by this result, we impose three requirements that deliver identification of the distribution of (T^*, X): (i) A voucher offer (weakly) incentivizes households to relocate;

TABLE I

TYPE PROBABILITIES AND CONDITIONAL EXPECTATION OF BASELINE VARIABLES.

	NN	NA	CN	CC	CA	AN	AA
				Type Definitions and Probabilities			
$(D^*(0), M^*(0))$	(0,0)	(0,1)	(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
$(D^*(1), M^*(1))$	(0,0)	(0,1)	(1,0)	(1,1)	(1,1)	(1,0)	(1,1)
Probability Point Estimate	0.259	0.253	0.196	0.068	0.201	0.012	0.010
s.e.	(0.014)	(0.014)	(0.013)	(0.024)	(0.023)	(0.004)	(0.003)
				Expected Value of Covariate Conditional on Types			
Household member	0.184	0.145	0.154	0.382	0.077	0.097	0.048
has a disability	(0.021)	(0.019)	(0.024)	(0.156)	(0.044)	(0.080)	(0.049)
No friends in the	0.347	0.392	0.422	0.568	0.409	0.361	0.775
neighborhood	(0.028)	(0.030)	(0.035)	(0.187)	(0.058)	(0.132)	(0.138)
Reports watching for	0.551	0.596	0.509	0.355	0.559	0.840	0.514
neighbor's children	(0.029)	(0.031)	(0.035)	(0.191)	(0.059)	(0.090)	(0.152)

Note: The first panel reports the support points of T^* and their respective estimated probabilities. The second panel reports estimates for the expectation of baseline variables conditioned on types. All estimates account for the person-level weight for the adult survey of the interim analyses. Standard errors are displayed in parentheses.

(ii) Moving to a low-poverty neighborhood (weakly) improves mental health; and (iii) A voucher offer affects mental health only through the relocation decision. Condition (ii) is motivated by extensive empirical work finding statistically significant estimates of the causal effect of relocation on mental health in the MTO experiment (Kling, Liebman, Katz, and Sanbonmatsu (2004)). Condition (ii) may nonetheless be violated, for instance, if there are individuals for whom the move causes social displacement that outweighs the benefits on mental health from moving to a lower poverty and safer neighborhood. Condition (iii) can be problematic for individuals who would have moved without a voucher, and for whom the voucher offer therefore represents a positive income shock. This concern is partly allayed by the estimated proportion of such individuals in our sample being below 2.5% (see Table I).

We impose these restrictions by letting $T^*(z) \equiv (D^*(z), M^*(z))$ with D^* and M^* describing how relocation and mental health respond to a voucher offer, and setting

$$\mu\big(D^*(1) \geq D^*(0), M^*(1) \geq M^*(0), \text{ and } D^*(1) = D^*(0) \Rightarrow M^*(1) = M^*(0)\big) = 1. \quad (7)$$

These restrictions limit the support of T^* to seven possible types. These types, displayed in the first panel of Table I, are characterized by the possible realizations of D^* and M^*, that is, by whether they are never takers, compliers, or always takers with regard to relocation and mental health status. For instance, types CN, CA, and CC are compliers with respect to relocation in that they relocate if and only if they are offered a voucher.

For any given response type t_0^*, we can write the probability that T^* equals t_0^* as the expectation in (4) evaluated at $\ell(T^*, X) = 1\{T^* = t_0^*\}$. Setting $\ell(t^*, X) = 1\{t^* = t_0^*\}$ in (5) yields a system of linear equations (in ν) that can be verified to have a solution for any t_0^* in the support of T^*.[4] Hence, the probability of each type is identified, and we can apply our asymptotically normal estimator based on (6) to estimate it; see Theorem 5.1.

[4]Let $\mathbf{T} = \{t_1, \dots, t_4\} = \{(0,0), (0,1), (1,0), (1,1)\}$ and set $\omega_0(t^*) = (1\{t^*(0) = t_1\}, \dots, 1\{t^*(0) = t_4\})'$ and $\omega_1(t^*) = (1\{t^*(1) = t_1\}, \dots, 1\{t^*(1) = t_4\})'$. For any t_0^*, we can find vectors s_0 and s_1 solving $s_0'\omega_0(t^*) + s_1'\omega_1(t^*) = 1\{t^* = t_0^*\}$ for all t^* by linear regression. Letting $(\nu(t_1, z), \dots, \nu(t_4, z)) = s_z'$ then solves (5).

Table I reports these estimates. While all types in our model occur with a strictly positive probability, the vast majority of households either do not relocate with a voucher offer (types NN, NA) or only relocate when offered a voucher (types CN, CC, CA).

Type CC is the only response type that experiences mental health improvement after relocation. Our results allow us to estimate how CC differs along baseline characteristics.[5] These estimates, reported in Table I, suggest that type CC is less attached to the baseline neighborhood and, therefore, less likely to experience social displacement after moving. They are second only to type AA in reporting not having friends in their neighborhood (at 56.8%) and the least likely to report watching for neighbor's children (at 35.5%). They are also the most likely to have a household member with a disability (at 38.2%), which we may expect to be associated with incurring greater benefits from moving to a low poverty neighborhood. While the point estimates are suggestive, we note that we cannot statistically reject that these baseline characteristics for type CC are the same as those of other types.

3.2. *Learning About Outcomes*

We next turn to estimating treatment effects in our model. A natural starting point is to examine the LATE of Imbens and Angrist (1994), which in our context equals

$$\mathrm{LATE} \equiv E_{Q_0}\big[Y^*\big(1, M^*(1)\big) - Y^*\big(0, M^*(0)\big) | T^* \in \{\mathrm{CN}, \mathrm{CC}, \mathrm{CA}\}\big].$$

The LATE informs us about the treatment effect of relocating for individuals who decide to relocate in response to a voucher offer. However, the LATE is a weighted average of causal effects across types with different mental health statuses and is, as a result, not suitable for assessing the mediating role of mental health. Specifically, the LATE aggregates

$$\begin{aligned} \mathrm{CDE}_0 &\equiv E_{Q_0}\big[Y^*(1,0) - Y^*(0,0) | T^* = \mathrm{CN}\big], \\ \mathrm{CDE}_1 &\equiv E_{Q_0}\big[Y^*(1,1) - Y^*(0,1) | T^* = \mathrm{CA}\big], \\ \mathrm{CTE} &\equiv E_{Q_0}\big[Y^*(1,1) - Y^*(0,0) | T^* = \mathrm{CC}\big]; \end{aligned} \tag{8}$$

that is, the LATE is a weighted average of the "controlled direct effects" of relocating while keeping mental health status constant (CDE_0 and CDE_1) and the "controlled total effect" of simultaneously relocating and improving mental health (CTE).

Because the marginal distribution of T^* is identified, the identification of CDE_0, CDE_1, and CTE reduces to the identification of expectations with the structure[6]

$$E_{Q_0}\big[\rho\big(Y^*(t)\big)\ell\big(T^*, X\big)\big]. \tag{9}$$

Our results imply that restriction (7) does not identify CDE_0, CDE_1, and CTE; see Corollary 4.4. We therefore introduce an "exogeneity of irrelevant mediator choices" (EIMC) assumption: Potential outcomes for high (resp., low) poverty neighborhood and poor (resp., good) mental health are conditionally independent of what mental health

[5]For example, since the probability $P(T^* = \mathrm{CC})$ is identified, we may set $\ell(T^*, X) = X1\{T^* = \mathrm{CC}\}/P(T^* = \mathrm{CC})$ in which case (4) becomes the mean of X conditional on being type CC.

[6]For example, $E[Y^*(t)|T^* = \mathrm{CN}]$ equals (9) with $\rho(Y^*(t)) = Y^*(t)$ and $\ell(T^*, X) = 1\{T^* = \mathrm{CN}\}/P(T^* = \mathrm{CN})$.

TABLE II

TREATMENT EFFECTS ESTIMATES.

	Sampled Adult is Employed	Sampled Adult not in Labor Force	Sampled Adult Earnings	Household Total Income
CDE_0	0.105	−0.025	1.116	1.067
	(0.066)	(0.071)	(1.366)	(2.154)
CDE_1	0.013	−0.094	1.280	3.884
	(0.083)	(0.075)	(1.788)	(2.341)
CTE	0.099	−0.386	3.323	1.537
	(0.107)	(0.121)	(2.383)	(3.239)
LATE	0.061	-0.113	1.412	2.258
	(0.052)	(0.051)	(1.093)	(1.569)

Note: Estimates for CDE_0, CDE_1, CTE (as in (8)), and LATE. Income measured in thousands of dollars per year. Estimates account for the person-level weight for the adult survey in interim analyses. Standard errors are displayed in parentheses.

would have been in a low (resp., high) poverty neighborhood. Formally, we use Assumption 2.2(iii) to require

$$Y^*(0,0) \perp\!\!\!\perp M^*(1)|D^*(1) > D^*(0), M^*(0) = 0, X,$$
$$Y^*(1,1) \perp\!\!\!\perp M^*(0)|D^*(1) > D^*(0), M^*(1) = 1, X.$$

EIMC introduces some homogeneity among compliers by implying that, conditional on X, the distribution of $Y^*(0,0)$ is the same for types CC and CN, and the distribution of $Y^*(1,1)$ is the same for types CC and CA.[7] This assumption is a slight weakening of a modification of the sequential ignorability assumption of Imai, Keele, and Yamamoto (2010) that was proposed by Yamamoto (2013) for instrumental variables models.

Our results imply EIMC and restriction (7) identify CDE_0, CDE_1, and CTE; see Theorem 4.3. More generally, our results imply (9) is identified if and only if

$$E_{Q_0}\left[\sum_{z\in\mathbf{Z}} 1\{T^*(z)=t\}\nu(z,X)|V^*(t),X\right] = E_{Q_0}\left[\ell(T^*,X)|V^*(t),X\right] \tag{10}$$

for some function ν of (Z,X) and $V^*(t)$ a known transformation of T^*.[8] Setting $\kappa(z,X)=\nu(z,X)/P(Z=z|X)$ for any ν solving (10), our results then imply (9) is identified by

$$E_{Q_0}\left[\rho\big(Y^*(t)\big)\ell\big(T^*,X\big)\right] = E_P\left[\rho(Y)1\{T=t\}\kappa(Z,X)\right]. \tag{11}$$

Table II reports treatment effects estimates based on (11) for different outcomes. LATE estimates reveal a statistically significant effect only on labor force participation. LATE estimates on income measures are both positive, but statistically insignificant. Decomposing these LATE estimates into CDE_0, CDE_1, and CTE (as defined in (8)) provides insights into the role of mental health as a mediator. CDE_0 and CDE_1, which report

[7] For example, in a threshold crossing model in which $D^*(z)=1\{z\geq\eta\}$ and $M^*(z)=1\{D^*(z)\geq\varepsilon\}$, EIMC is implied by $Y^*(t)$ being independent of ε conditionally on η and ε and η being independent.

[8] Specifically, we set $V^*(t)=0$ when $t=(0,0)$ and $T^*\in\{\mathrm{CN},\mathrm{CC}\}$, $V^*(t)=1$ when $t=(1,1)$ and $T^*\in\{\mathrm{CA},\mathrm{CC}\}$, and $V^*(t)=T^*$ for all other values of t and T^*.

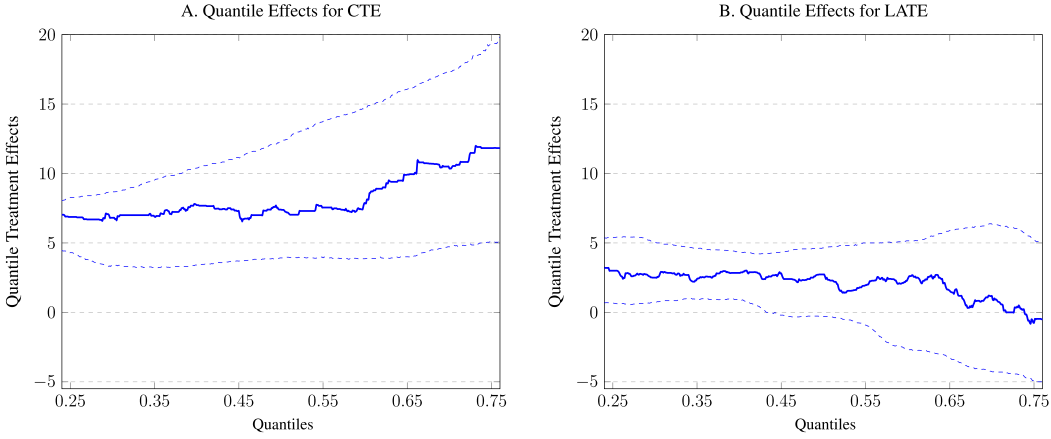

FIGURE 1.—Quantile Treatment Effects for Total Household Income. *Note*: Graph A: Difference in quantiles of $Y^*(1,1)$ and $Y^*(0,0)$ conditional on $T^* = CC$. Graph B: Difference in the quantiles of $Y^*(1, M^*(1))$ and $Y^*(0, M^*(0))$ conditional on $T^* \in \{\text{CA}, \text{CN}, \text{CC}\}$. Income measured in thousands of dollars per year. Estimates account for the person-level weight for the adult survey in interim analyses. Dashed lines are 95% pointwise equitailed bootstrap confidence regions.

treatment effects for types CN and CA, suggest a positive effect on labor force participation but are statistically insignificant. CTE for labor force participation, which reports treatment effects for type CC, is in contrast statistically significant and an order of magnitude larger than CDE_0 and CDE_1. Because CC is the only complier type whose mental health changes after relocation, these results lend credence to the conjecture by LLKD+ (2008) that improved mental health is a mediator through which relocation affects labor force participation.

While the positive CTE and LATE estimates for income measures are not statistically significant, we find significant Quantile Treatment Effects (QTEs) for type CC on Household Total Income. Figure 1 reports these estimates with 95% (pointwise) bootstrap equitailed confidence regions. QTEs corresponding to both relocating and improving mental health (CTE) are positive and statistically significant across the quantiles we examine. Reflecting the low proportion of type CC, the LATE QTEs exhibit mixed findings, being statistically significant for lower quantiles only.

4. IDENTIFICATION

In this section, we characterize point identification and provide a recipe for estimation.

4.1. *Two Key Lemmas*

We begin by introducing two lemmas that play a fundamental role in our analysis.

LEMMA 4.1: *If Assumptions 2.1 and 2.2 hold, then Θ_0 is convex and there is a $\bar{Q} \in \Theta_0$ such that all $Q \in \Theta_0$ are absolutely continuous with respect to $\bar{Q}$.*

Lemma 4.1 shows there is a distribution $\bar{Q}$ in the identified set Θ_0 that is "larger" than any other distribution in Θ_0. By "larger," we mean that the support of any distribution in the identified set must be contained in the support of $\bar{Q}$. There may be multiple distributions $\bar{Q}$ satisfying the conclusion of Lemma 4.1. However, such distributions are equivalent in the sense that they must assign zero probability to the same sets.

Our second lemma is the cornerstone of our identification analysis. In order to formally state this key result, we introduce a linear operator Υ that maps functions of (Y, T, Z, X) to functions of (Y^*, T^*, X). Specifically, for any $f \in L^1(P)$ we set

$$\Upsilon(f) \equiv \sum_{t \in \mathbf{T}} E_{P_{Z|X}}\big[f\big(Y^*(t), t, Z, X\big)1\big\{T^*(Z) = t\big\}\big], \tag{12}$$

where $P_{Z|X}$ denotes the conditional distribution of Z given X and the notation $E_{P_{Z|X}}$ emphasizes the expectation is taken with respect to Z while (Y^*, T^*, X) are kept fixed. Under Assumption 2.1, it is possible to show that $\Upsilon(f)$ in fact satisfies

$$\Upsilon(f) = E_Q\big[f(Y, T, Z, X)|Y^*, T^*, X\big] \tag{13}$$

for any Q in the identified set Θ_0, that is, Υ maps functions of observables into functions of unobservables by taking conditional expectations given (Y^*, T^*, X).

The next lemma combines the map Υ with the distribution $\bar{Q}$ to obtain a sufficient condition for the expectation of $\ell(Y^*, T^*, X)$ to be identified for a given function ℓ.

LEMMA 4.2: *Let Assumptions 2.1 and 2.2 hold, and ℓ satisfy $\bar{Q}(\Upsilon(\kappa) = \ell) = 1$ for some $\kappa \in L^1(P)$. Then it follows that $\ell \in L^1(Q)$ for all $Q \in \Theta_0$ and in addition*

$$E_Q\big[\ell\big(Y^*, T^*, X\big)\big] = E_P\big[\kappa(Y, T, Z, X)\big]. \tag{14}$$

Lemma 4.2 follows from noting that the conditions imposed on κ and result (13) imply

$$E_Q\big[\kappa(Y, T, Z, X)|Y^*, T^*, X\big] = \ell\big(Y^*, T^*, X\big)$$

for any $Q \in \Theta_0$ and, therefore, result (14) is immediate by the law of iterated expectations. The main implication of Lemma 4.2 is a recipe for identification and estimation. In particular, Lemma 4.2 suggests estimating the expectation of $\ell(Y^*, T^*, X)$ by employing sample moments based on an estimator of a function κ satisfying $\Upsilon(\kappa) = \ell$. To this end, it is fruitful to rely on our next corollary, which represents κ in terms of the density $\pi \equiv dP_{Z|X}/d\mu_{Z|X}$.

COROLLARY 4.1: *Let Assumptions 2.1 and 2.2 hold, and $\nu \in L^1(P)$ satisfy*

$$\mu\bigg(\sum_{t \in \mathbf{T}} E_{\mu_{Z|X}}\big[\nu\big(Y^*(t), t, Z, X\big)1\big\{T^*(Z) = t\big\}\big] = \ell\big(Y^*, T^*, X\big)\bigg) = 1. \tag{15}$$

If $\mu(\pi(Z, X) > \delta) = 1$ for some $\delta > 0$, then the function $\kappa \equiv \nu/\pi$ satisfies $\kappa \in L^1(P)$ and $\bar{Q}(\Upsilon(\kappa) = \ell) = 1$ and, therefore, $E_{Q_0}[\ell(Y^*, T^*, X)] = E_P[\kappa(Y, T, Z, X)]$.

Corollary 4.1 shows that we may find the desired κ by taking the ratio of a function ν satisfying (15) and the density $\pi \equiv dP_{Z|X}/d\mu_{Z|X}$. In the majority of our examples, μ can be chosen to be known and ν can be computed analytically or numerically. Hence, in these cases, κ is known up to π. Our estimators in Section 5 heavily rely on this observation.

REMARK 4.1: Example 2.1, based on Rosenbaum and Rubin (1983), helps illustrate Corollary 4.1. Recall that we set $\mu(T^*(Z)=Z)=1$ and $\ell(Y^*, T^*, X) = Y^*(1) - Y^*(0)$. Corollary 4.1 suggests identifying the ATE by finding a ν satisfying (15). To this end, it is convenient to set μ to satisfy $\mu(Z=0|X) = \mu(Z=1|X) = 1/2$, which implies $\nu(Y, T, Z, X) = 2Y(1\{Z=1\} - 1\{Z=0\})$ solves (15) and the density $\pi \equiv dP_{Z|X}/d\mu_{Z|X}$ equals $\pi(z, X) = 2P(Z=z|X)$. Corollary 4.1 and computing $\kappa \equiv \nu/\pi$ then yield

$$E_{Q_0}\big[Y^*(1) - Y^*(0)\big] = E_P\big[\kappa(Y, T, Z, X)\big] = E_P\left[\frac{Y1\{Z=1\}}{P(Z=1|X)} - \frac{Y1\{Z=0\}}{P(Z=0|X)}\right],$$

which recovers the propensity score reweighing moment for identifying the ATE. Similarly, applying Corollary 4.1 to the model in Imbens and Angrist (1994) recovers the κ-weights identifying equations of Abadie (2003). In both cases, the distribution of Y^* under μ can be left unspecified and the distribution of X under μ set to equal the distribution under P. More generally, in models that only restrict the support of T^*, it is often computationally convenient to set μ so that (T^*, Z, X) are mutually independent and T^* and Z are uniformly distributed. We employ this approach below when revisiting Examples 2.4 and 2.5.

4.2. *Main Result*

Lemma 4.2 shows that a sufficient condition for identifying the expectation

$$\lambda_{Q_0} \equiv E_{Q_0}\big[\ell\big(Y^*, T^*, X\big)\big] \tag{16}$$

is that there be a function κ satisfying $\Upsilon(\kappa) = \ell$. Moreover, Lemma 4.2 also suggests an estimation strategy for λ_{Q_0}. We next show this sufficient condition for identification is close to necessary and, as a result, the estimation strategy applies whenever λ_{Q_0} is identified.

Concretely, we will show λ_{Q_0} is identified *if and only if* the function ℓ is in the closure of the set of functions that equal $\Upsilon(\kappa)$ for some κ—notice the contrast with Lemma 4.2, which requires ℓ to *equal* $\Upsilon(\kappa)$ for some κ. This result implies that for λ_{Q_0} to be identified, ℓ must be the limit of functions whose expectations are identified through Lemma 4.2. This characterization is constructive in that it suggests estimating λ_{Q_0} by employing the sample moments of a sequence of functions $\{\kappa_j\}$ for which $\Upsilon(\kappa_j)$ converges to ℓ. Formalizing this discussion, however, first requires clarifying the sense (i.e., topology) in which we mean converge, closure, and limit. To this end, we introduce one additional assumption.

ASSUMPTION 4.1: (*i*) *$\ell \in L^1(Q)$ for all $Q \in \Theta_0$*; (*ii*) *The density $dQ/d\mu$ belongs to the interior of $\mathcal{Q}$ in* $\mathbf{Q}$ *for some $Q \in \Theta_0$.*

Assumption 4.1(i) requires ℓ be integrable with respect to every $Q \in \Theta_0$. This condition can be ensured by imposing suitable regularity conditions through the choice of $\mathcal{Q}$ in Assumption 2.2(iii). Assumption 4.1(ii) is needed to show the necessity, but not the sufficiency, of our conditions for point identification of λ_{Q_0}. This requirement is automatically satisfied in applications in which $\mathcal{Q}$ equals $\mathbf{Q}$; see Corollaries 4.2, 4.3, and 4.4 below.

As we discussed, identification of λ_{Q_0} turns on whether ℓ belongs to the closure of the set of functions that equal $\Upsilon(\kappa)$ for some κ. The latter set is the range of Υ, denoted by

$$\mathcal{R} \equiv \left\{f \in \bigcap_{Q \in \Theta_0} L^1(Q) : f = \Upsilon(\kappa) \text{ for some } \kappa \in L^1(P)\right\}.$$

To introduce the topology determining the closure of $\mathcal{R}$, we define $\langle f, g\rangle_Q \equiv \int fg\,dQ$, let Q_V denote the marginal distribution of a random variable V under Q, and define the set

$$\mathcal{S}_Q \equiv \left\{ s \in L^\infty(Q_{Y^*T^*X}) \text{ such that } s\frac{dQ}{d\mu} \in \mathbf{Q} \right\},$$

where $L^\infty(Q_{Y^*T^*X})$ is the set of functions of (Y^*, T^*, X) that are bounded with probability one under Q. Additionally, we let span$\{A\}$ denote the linear span of a set A, and introduce a vector space $\mathcal{B}$ of linear functionals defined on $\bigcap_{Q\in\Theta_0} L^1(Q)$ by setting

$$\mathcal{B} \equiv \text{span}\left\{ B : \bigcap_{Q\in\Theta_0} L^1(Q) \to \mathbf{R} \text{ such that } B = \langle \cdot, s\rangle_Q \text{ for some } s \in \mathcal{S}_Q, Q \in \Theta_0 \right\}.$$

Crucially, by Assumption 4.1(i) and Lemma 4.2, ℓ and all functions $f \in \mathcal{R}$ belong to the domain of all the linear functionals $B \in \mathcal{B}$. Using this observation, we define the topology τ on $\mathcal{R} \cup \{\ell\}$ to be the weak topology generated by the functionals $B \in \mathcal{B}$, that is, a sequence $\{f_j\}$ in $\mathcal{R}$ converges to ℓ under τ if and only if $B(f_j) \to B(\ell)$ for all $B \in \mathcal{B}$.

The next theorem is our main identification result.

THEOREM 4.1: *If Assumptions 2.1, 2.2, and 4.1 hold, then it follows that λ_{Q_0} is identified if and only if ℓ belongs to the τ-closure of $\mathcal{R}$.*

Intuitively, Theorem 4.1 shows λ_{Q_0} is identified if and only if there is a sequence $\{f_j\} \subseteq \mathcal{R}$ converging to ℓ under τ. To gain insight into why this condition is sufficient for identification, let $\mathbf{1}$ be the function that is constant at one, set $B_{Q_0} \equiv \langle \cdot, \mathbf{1}\rangle_{Q_0}$, and note

$$\lambda_{Q_0} = \langle \ell, \mathbf{1}\rangle_{Q_0} = B_{Q_0}(\ell). \tag{17}$$

Since $f_j \in \mathcal{R}$ implies $f_j = \Upsilon(\kappa_j)$ for some function κ_j of (Y, T, Z, X), it follows from $\{f_j\}$ converging to ℓ in the τ topology that there is a sequence $\{\kappa_j\}$ satisfying

$$B_{Q_0}(\ell) = \lim_{j\to\infty} B_{Q_0}(f_j) = \lim_{j\to\infty} \langle \Upsilon(\kappa_j), \mathbf{1}\rangle_{Q_0} = \lim_{j\to\infty} E_P\big[\kappa_j(Y, T, Z, X)\big], \tag{18}$$

where the final equality follows from Lemma 4.2. Results (17) and (18) not only establish the identification of λ_{Q_0}, but also suggest an estimation strategy: Simply employ sample moments based on estimates of the functions $\{\kappa_j\}$. As illustrated by Remark 4.1, the sequence $\{\kappa_j\}$ may be intuitively understood as a generalization of propensity score reweighing or Abadie's kappa to the large class of causal models that we study.

The conclusion that the existence of the desired $\{\kappa_j\}$ is necessary for the identification of λ_{Q_0} is more surprising.[9] Importantly, necessity implies it is without loss of generality to estimate λ_{Q_0} by employing an estimation strategy based on (18). The next corollary illustrates how Theorem 4.1 can also characterize $\{\kappa_j\}$, which is helpful for estimation.

COROLLARY 4.2: *Let Assumptions 2.1, 2.2 hold with $\mathcal{Q} = \mathbf{Q} = L^\infty(\mu)$, $\mu \ll \bar{Q}$ with $d\mu/d\bar{Q}$ bounded, and $\ell \in L^1(\mu_{Y^*T^*X})$. Then, it follows that*:

[9]Theorem 4.1 only implies the existence of a net, which is a generalized sequence whose index set is a possibly uncountable directed set, rather than the natural numbers. However, we discuss sequences for ease of exposition.

(i) *λ_{Q_0} is identified if and only if $\lim_{j\to\infty}\|\ell-\Upsilon(\kappa_j)\|_{\mu,1}=0$ for some $\{\kappa_j\}\subseteq L^1(P)$. Moreover, any such sequence $\{\kappa_j\}$ satisfies $\lambda_{Q_0}=\lim_{j\to\infty}E_P[\kappa_j(Y,T,Z,X)]$.*
(ii) *Suppose in addition that $\mu(\pi(Z,X)>\delta)=1$ for some $\delta>0$ and $\pi=dP_{Z|X}/d\mu_{Z|X}$. Then λ_{Q_0} is identified if and only if there is a sequence $\{\nu_j\}\subseteq L^1(P)$ satisfying*

$$\lim_{j\to\infty}E_\mu\left[\left|\ell(Y^*,T^*,X)-\sum_{t\in\mathbf{T}}E_{\mu_{Z|X}}\big[\nu_j\big(Y^*(t),t,Z,X\big)1\big\{T^*(Z)=t\big\}\big]\right|\right]=0. \quad (19)$$

Moreover, for any such $\{\nu_j\}$, $\kappa_j\equiv\nu_j/\pi$ satisfies $\lambda_{Q_0}=\lim_{j\to\infty}E_P[\kappa_j(Y,T,Z,X)]$.

Corollary 4.2 specializes Theorem 4.1 by imposing that Assumption 2.2(iii) only require the density of Q_0 to be bounded. The corollary also assumes $\mu\ll\bar{Q}$, which ensures that every response type allowed for by μ has positive probability under $\bar{Q}$.[10] Corollary 4.2(i) shows λ_{Q_0} is identified if and only if ℓ is in the closure of $\mathcal{R}$ under $\|\cdot\|_{\mu,1}$. Corollary 4.2(ii) provides conditions under which we can set $\kappa_j=\nu_j/\pi$ for any $\{\nu_j\}$ satisfying (19) and $\pi\equiv dP_{Z|X}/d\mu_{Z|X}$. This result has two important implications. First, Corollary 4.2(ii) provides a characterization of $\{\kappa_j\}$ that we use in estimation. Second, it allows us to assess whether the restrictions of our model point identify λ_{Q_0}; see Example 2.5 below.

REMARK 4.2: Our main identification results do not fundamentally rely on T being discrete. Under a mild modification to the proofs, Theorem 4.1 and Corollary 4.2(i) hold when T is continuous provided that for any $f(Y,T,Z,X)$ we understand $\Upsilon(f)$ to be

$$\Upsilon(f)\equiv E_{P_{Z|X}}\big[f\big(Y^*\big(T^*(Z)\big),T^*(Z),Z,X\big)\big]. \quad (20)$$

Note that this definition for Υ coincides with the one in (12) when T is discrete, but it remains valid when T is continuous as well. Some of our forthcoming results, such as Theorem 4.3 below, rely more essentially on T being discrete.

4.3. *Special Case: Types*

Features of the distribution of (T^*,X) are identified by specializing our analysis to expectations of functions of only (T^*,X), that is, to parameters with the structure

$$\lambda_{Q_0}\equiv E_{Q_0}\big[\ell\big(T^*,X\big)\big]. \quad (21)$$

While Theorem 4.1 applies to these parameters, the restriction that ℓ depend only on (T^*,X) enables sharper conclusions. In particular, we will show λ_{Q_0} is identified if and only if it is identified from the distribution of (T,Z,X). To this end, we define

$$\bar{Q}^{\mathrm{it}}\equiv\bar{Q}_{Y^*|X}\bar{Q}_{T^*ZX},$$

which shares the same marginal distributions for (Y^*,X) and (T^*,X) as $\bar{Q}$, but is such that Y^* is independent of T^* conditionally on X. Given this notation, we impose the following.

[10] For example, in Imbens and Angrist (1994), the requirement $\mu\ll\bar{Q}$ rules out compliers having probability one.

ASSUMPTION 4.2: *(i)* $\bar{Q}^{\text{it}} \ll \bar{Q}$; *(ii)* $(d\bar{Q}^{\text{it}}_{Y^*T^*X}/d\bar{Q}_{Y^*T^*X})s \in \mathcal{S}_{\bar{Q}}$ *for all* $s \in \mathcal{S}_{\bar{Q}} \cap L^{\infty}(\bar{Q}_{T^*X})$; *(iii)* $(dQ_{T^*X}/d\bar{Q}_{T^*X})s \in \mathcal{S}_{\bar{Q}}$ *for all* $Q \in \Theta_0$ *and* $s \in L^{\infty}(Q_{T^*X}) \cap \mathcal{S}_Q$; *(iv) For any* $Q \in \Theta_0$ *and* $s \in \mathcal{S}_Q$, *we have that* $E_Q[s(Y^*, T^*, X)|T^*, X] \in \mathcal{S}_Q$.

Assumption 4.2(i) requires the support of Y^* conditional on (T^*, X) under $\bar{Q}$ to not depend on T^*. Assumptions 4.2(ii) and 4.2(iii) impose restrictions on the densities $d\bar{Q}^{\text{it}}/d\bar{Q}$ and $dQ_{T^*X}/d\bar{Q}_{T^*X}$ (for $Q \in \Theta_0$), while Assumption 4.2(iv) requires that conditional expectations of functions in $\mathcal{S}_Q$ belong to $\mathcal{S}_Q$ as well. Assumptions 4.2(ii)–(iv) can in many applications be verified by appropriately selecting $\mathcal{Q}$ and $\mathbf{Q}$ in Assumption 2.2(iii).

The next result characterizes identification of features of the distribution of (T^*, X).

THEOREM 4.2: *Let Assumptions 2.1, 2.2, 4.1, 4.2 hold,* λ_{Q_0} *be as in (21), and define*

$$\mathcal{R}_T \equiv \left\{ f \in \bigcap_{Q \in \Theta_0} L^1(Q) : f = \Upsilon(\kappa) \text{ for some } \kappa \in L^1(P_{TZX}) \right\}.$$

Then it follows that λ_{Q_0} *is identified if and only if* ℓ *belongs to the* τ*-closure of* $\mathcal{R}_T$.

Theorem 4.2 implies that identification of λ_{Q_0} turns on whether ℓ can be approximated by a sequence $\{\Upsilon(\kappa_j)\}$ with $\{\kappa_j\}$ depending only on (T, Z, X)—a conclusion that contrasts with Theorem 4.1, in which κ_j depends on (Y, T, Z, X). When such an approximating sequence exists, λ_{Q_0} equals the limit of the expectations of the corresponding κ_j.

The next result highlights the implications of Theorem 4.2 when instruments are discrete.

COROLLARY 4.3: *Let Assumption 2.1, 2.2 hold,* $\mathcal{Q} = \mathbf{Q} = L^1(\mu)$, $\bar{Q}^{\text{it}} \ll \bar{Q}$ *with* $d\bar{Q}^{\text{it}}/d\bar{Q}$ *bounded,* $\ell \in L^1(\bar{Q}_{T^*X})$, Z *be discrete,* $P(Z = z|X) \geq \varepsilon > 0$ *a.s. for all* $z \in \mathbf{Z}$, *and* $\bar{Q}(T^* = t^*|X) \geq \varepsilon > 0$ *a.s. for any* $t^* \in \mathbf{T}^*$ *with* $\bar{Q}(T^* = t^*) > 0$. *Then*:

(i) $\lambda_{Q_0} \equiv E_{Q_0}[\ell(T^*, X)]$ *is identified if and only if* $\bar{Q}(\Upsilon(\kappa) = \ell) = 1$ *for some* $\kappa \in L^1(P_{TZX})$. *Moreover, any such* κ *satisfies* $\lambda_{Q_0} = E_P[\kappa(T, Z, X)]$.

(ii) *Suppose in addition that* $\mu \ll \bar{Q}$ *and* $\mu(\pi(Z, X) > \delta) = 1$ *for some* $\delta > 0$ *and* $\pi \equiv dP_{Z|X}/d\mu_{Z|X}$. *Then* λ_{Q_0} *is identified if and only if there is a* $\nu \in L^1(P_{TZX})$ *satisfying*

$$\mu\left(\ell(T^*, X) = \sum_{t \in \mathbf{T}} E_{\mu_{Z|X}}\left[\nu(t, Z, X) 1\{T^*(Z) = t\}\right]\right) = 1. \tag{22}$$

Moreover, for any such function ν, $\kappa \equiv \nu/\pi$ *satisfies* $\lambda_{Q_0} = E_P[\kappa(T, Z, X)]$.

Corollary 4.3(i) specializes Theorem 4.2 to the case of discrete instruments and no identifying assumptions being imposed through $\mathcal{Q}$. In this context, Corollary 4.3(i) provides a converse to Lemma 4.2 by showing λ_{Q_0} is identified if and only if ℓ equals $\Upsilon(\kappa)$ for some function κ of (T, Z, X). Corollary 4.3(ii) parallels Corollary 4.2 in providing conditions that are useful for assessing whether λ_{Q_0} is identified and estimating it if it is. Together with Corollary 4.4 below, Corollary 4.3(ii) provides conditions under which the sufficient condition for identification derived by Heckman and Pinto (2018) for discrete instrument applications (such as Example 2.4) is also necessary.[11]

[11]Specifically, in a setting with no covariates ℓ may be interpreted as a vector. Theorem T-2 in Heckman and Pinto (2018) then shows $E_{Q_0}[\ell(T^*)]$ is identified when ℓ is orthogonal to the null space of a matrix B. In

4.3.1. *Examples Revisited*

We illustrate our results by revisiting Examples 2.4 and 2.5. For calculations, it is helpful to note $\kappa = \nu/\pi$ is invariant to scaling μ, and hence μ need not be a probability measure.

Example 2.4 (cont.) In this example, Z and T^* were discrete and μ imposed $\mu(T^* \in \mathbf{R}^*) = 1$ for a set $\mathbf{R}^* \equiv \{t_1^*, \dots, t_r^*\}$. Let $\mathbf{Z} \equiv \{z_1, \dots, z_q\}$ denote the support of Z and $\omega_j(t^*) \equiv (1\{t^*(z_j) = t_1\}, \dots, 1\{t^*(z_j) = t_d\})'$. As in Remark 4.1, we pick μ with Z independent of X under μ. Corollary 4.3(ii) implies the expectation of $\ell(T^*, X)$ is identified if and only if

$$\min_{\{s_j\}_{j=1}^q \subset \mathbf{R}^d} \sum_{i=1}^r \left(\ell\left(t_i^*, X\right) - \sum_{j=1}^q s_j' \omega_j\left(t_i^*\right) \right)^2 = 0 \tag{23}$$

with probability one (over X). Moreover, provided (23) holds, we can find a function κ of (T, Z, X) whose expectation equals the expectation of $\ell(T^*, X)$ by setting $(\kappa(t_1, z_j, X), \dots, \kappa(t_d, z_j, X)) \equiv (s_j(X)/P(Z = z_j|X))'$ for any $(s_1(X), \dots, s_q(X))$ minimizing (23). Specializing (23) to Kline and Walters (2016) implies the distribution of (T^*, X) is identified in that context. More generally, we highlight that identifying features of the distribution of (T^*, X) reduces to a least squares problem when Z is discrete.

Example 2.5 (cont.) In this example, $Z = (C, W)$ with $C \in \{0, 1\}$, $W \in \mathbf{R}$, and μ required that $T^*(c, w)$ be increasing in c and decreasing in w. Assuming W is continuously distributed with compact support $[\underline{\mathbf{w}}, \overline{\mathbf{w}}]$, it is possible to show there are (K_0^*, K_1^*) with

$$T^*(c, w) = \sum_{i=0}^1 1\left\{c = i, K_i^* > w\right\} \tag{24}$$

and $K_i^* \in [\underline{\mathbf{w}}, \overline{\mathbf{w}}] \cup \{\infty\}$. We study identification of the distribution of (K_0^*, K_1^*, X) and note the restriction that $T^*(c, w)$ be increasing in c is equivalent to $\mu(K_1^* \geq K_0^*) = 1$. It is convenient to let K_i^* be continuously distributed on $(\underline{\mathbf{w}}, \overline{\mathbf{w}}]$ under μ, but allow μ to assign positive mass to $\{\underline{\mathbf{w}}\}$ and $\{\infty\}$, that is, to assign positive probability to always takers and never takers. Under conditions paralleling those in Corollary 4.2(ii), Theorem 4.2 then implies that the expectation of a function ℓ of (K_0^*, K_1^*, X) is identified if and only if

$$\lim_{j \to \infty} E_\mu \left[\left| \sum_{i=0}^1 \int_{K_i^*}^{\bar{\mathbf{w}}} \nu_j(0, i, w, X)\, dw + \int_{\underline{\mathbf{w}}}^{K_i^*} \nu_j(1, i, w, X)\, dw - \ell\left(K_0^*, K_1^*, X\right) \right| \right] = 0 \tag{25}$$

for some sequence $\{\nu_j\}$ of functions of (T, C, W, X). Moreover, provided condition (25) holds, the expectation of $\ell(K_0^*, K_1^*, X)$ equals the limit of the expectations of the functions

$$\kappa_j(t, c, w, X) \equiv \frac{\nu_j(t, c, w, X)}{f_{W|CX}(w|c, X) P(C = c|X)},$$

where $f_{W|CX}$ denotes the density of W given (C, X). More generally, Theorem 4.2 implies the expectation of $\ell(K_0^*, K_1^*, X)$ is identified if and only if ℓ belongs to the closure of $\mathcal{T} \equiv \{f : f(K_0^*, K_1^*, X) = g_0(K_0^*, X) + g_1(K_1^*, X)$ for some $g_0, g_1\}$ under $\|\cdot\|_{\mu,1}$. For instance, the probability of the event $\{K_1^* > a_1, K_0^* \leq a_0\}$ is identified if and only if $a_0 \geq a_1$.

contrast, in this context (22) may be understood as the equivalent condition that ℓ belong to the range of the adjoint B'.

4.4. *Special Case: Outcomes*

We conclude our identification analysis by focusing on parameters with the structure

$$\lambda_{Q_0} \equiv E_{Q_0}\big[\rho\big(Y^*(t), X\big)\ell\big(T^*, X\big)\big], \tag{26}$$

for some $t \in \mathbf{T}$ and functions ρ and ℓ. These parameters include features of the conditional distribution of a potential outcome given types and covariates as a special case. Intuitively, identification of these parameters can only be possible from the distribution of observations for which treatment T equals t. To formalize this intuition, we first define

$$\bar{Q}^{\mathrm{io}} \equiv \bar{Q}_{Y^*(t_1)|X} \cdots \bar{Q}_{Y^*(t_d)|X}\bar{Q}_{T^*ZX};$$

that is, $\bar{Q}^{\mathrm{io}}$ has the same marginal distributions for $(Y^*(t), X)$ and (T^*, X) as $\bar{Q}$, but is such that all coordinates of Y^* and T^* are mutually independent conditional on X. We also define

$$\phi_{\bar{Q},\rho}\big(Y^*(t), X\big) \equiv \frac{\rho\big(Y^*(t), X\big) - E_{\bar{Q}}\big[\rho\big(Y^*(t), X\big)|X\big]}{\mathrm{Var}_{\bar{Q}}\big\{\rho\big(Y^*(t), X\big)|X\big\}},$$

where we set $\phi_{\bar{Q},\rho}(Y^*(t), X) = 0$ when $\mathrm{Var}_{\bar{Q}}\{\rho(Y^*(t), X)|X\} = 0$.

Given the introduced notation, we impose the following assumptions.

ASSUMPTION 4.3: *$\rho \in L^\infty(Q)$ and $\ell \in L^1(Q)$ for all $Q \in \Theta_0$.*

ASSUMPTION 4.4: *(i) $E_Q[\rho(Y^*(t), X)|T^*, X] \in \mathcal{S}_Q$ and $E_Q[s(Y^*, T^*, X)|T^*, X] \in \mathcal{S}_Q$ for any $Q \in \Theta_0$ and $s \in \mathcal{S}_Q$; (ii) $\bar{Q}^{\mathrm{io}} \ll \bar{Q}$ and $d\bar{Q}^{\mathrm{io}}/d\bar{Q} \in L^\infty(\bar{Q})$; (iii) $\phi_{\bar{Q},\rho} \in L^\infty(\bar{Q})$, $\mathrm{Var}_{\bar{Q}}\{\rho(Y^*(t), X)|X\} > 0$ a.s. under $\bar{Q}$, and $s(d\bar{Q}^{\mathrm{io}}_{Y^*T^*X}/d\bar{Q}_{Y^*T^*X})\phi_{\bar{Q},\rho} \in \mathcal{S}_{\bar{Q}}$ for all $s \in L^\infty(\bar{Q}_{T^*X}) \cap \mathcal{S}_{\bar{Q}}$; (iv) $(dQ_{T^*X}/d\bar{Q}_{T^*X})s \in \mathcal{S}_{\bar{Q}}$ for all $Q \in \Theta_0$ and $s \in L^\infty(Q_{T^*X}) \cap \mathcal{S}_Q$.*

Assumption 4.3 ensures $\ell\rho$ is integrable for all $Q \in \Theta_0$ and, therefore, that λ_{Q_0} is well-defined. Assumption 4.4 is similar in spirit to the conditions we imposed in Assumption 4.2 for studying features of the distribution of (T^*, X). Specifically, Assumptions 4.4(i)(iv) parallel Assumptions 4.2(ii)–(iv), while Assumption 4.4(ii) imposes a key support requirement that parallels Assumption 4.2(i). Finally, Assumption 4.4(iii) ensures that ρ is not constant in $Y^*(t)$, and that therefore (26) does not fall within the framework of Section 4.3.

The next result shows parameters with the structure in (26) are identified if and only if they are identified from the distribution of observations with treatment T equal to t.

THEOREM 4.3: *Let Assumptions 2.1, 2.2, 4.1(ii), 4.3, 4.4 hold, λ_{Q_0} be as in (26), define $L^1(P_{tZX}) \equiv \{f \in L^1(P) : f(T, Z, X) = 1\{T = t\}g(Z, X)$ for some $g \in L^1(P)\}$, and*

$$\mathcal{R}_t \equiv \left\{f \in \bigcap_{Q\in\Theta_0} L^1(Q) : f = \Upsilon(\kappa) \text{ for some } \kappa \in L^1(P_{tZX})\right\}.$$

Then it follows that λ_{Q_0} is identified if and only if ℓ belongs to the τ-closure of $\mathcal{R}_t$.

We emphasize the contrast with Theorem 4.2, which showed identification of the expectation of $\ell(T^*, X)$ is equivalent to ℓ being in the τ-closure of $\mathcal{R}_T$ (instead of $\mathcal{R}_t$ in

Theorem 4.3). In particular, since $\mathcal{R}_t \subseteq \mathcal{R}_T$, it follows that identification of the expectation of $\rho(Y^*(t), X)\ell(T^*, X)$ implies the identification of the expectation of $\ell(T^*, X)$. More generally, since ℓ belonging to $\mathcal{R}_t$ does not depend on the choice of ρ, the identification of λ_{Q_0} for *some* ρ implies that λ_{Q_0} is in fact identified for *all* suitable ρ. Our next corollary illustrates these implications in the case of discrete instruments.

COROLLARY 4.4: *Let Assumptions 2.1, 2.2 hold, $\mathcal{Q} = \mathbf{Q} = L^1(\mu)$, $\bar{Q}^{\text{io}} \ll \bar{Q}$ with $d\bar{Q}^{\text{io}}/d\bar{Q}$ and ρ bounded, $\ell \in L^1(\bar{Q}_{T^*X})$, and $\operatorname{Var}_{\bar{Q}}\{\rho(Y^*(t), X)|X\} \geq \varepsilon > 0$ a.s. If Z is discrete, $P(Z = z|X) \geq \varepsilon > 0$ a.s. for all $z \in \mathbf{Z}$, and $\bar{Q}(T^* = t^*|X) \geq \varepsilon > 0$ a.s. for any $t^* \in \mathbf{T}^*$ with $\bar{Q}(T^* = t^*) > 0$, then the following are equivalent*:

(i) *$E_{Q_0}[\rho(Y^*(t), X)\ell(T^*, X)]$ is identified.*

(ii) *$E_{Q_0}[f(Y^*(t), X)\ell(T^*, X)]$ is identified for any bounded f.*

(iii) *$\bar{Q}(\Upsilon(\kappa) = \ell) = 1$ for some $\kappa(T, Z, X) = 1\{T = t\}g(Z, X)$ with $g \in L^1(P_{ZX})$ and, therefore, $E_{Q_0}[f(Y^*(t), X)\ell(T^*, X)] = E_P[f(Y, X)\kappa(T, Z, X)]$ for any bounded f.*

Through the equivalence of (i) and (ii), Corollary 4.4 formalizes that identification of λ_{Q_0} for some ρ implies identification of λ_{Q_0} for all ρ. Corollary 4.4 also shows identification of λ_{Q_0} requires that $\ell = \Upsilon(\kappa)$ for some κ that only employs observations with treatment T equal to t. Moreover, paralleling Corollaries 4.2 and 4.3, it is possible to show that we may set $\kappa(T, Z, X) = 1\{T = t\}\nu(Z, X)/\pi(Z, X)$ with $\pi \equiv dP_{Z|X}/d\mu_{Z|X}$ and any ν satisfying

$$\ell(T^*, X) = E_{\mu_{Z|X}}\big[\nu(Z, X)1\big\{T^*(Z) = t\big\}\big]. \tag{27}$$

REMARK 4.3: Our results also imply necessary and sufficient conditions for the identification of parameters that, for some known ρ and ℓ, have the structure

$$\frac{E_{Q_0}\big[\rho\big(Y^*(t), X\big)\ell\big(T^*, X\big)\big]}{E_{Q_0}\big[\ell\big(T^*, X\big)\big]}. \tag{28}$$

These parameters include, for instance, the expectation of $Y^*(t)$ conditional on T^* belonging to a set. By Theorems 4.2 and 4.3, if ℓ belongs to the τ-closure of $\mathcal{R}_t$, then the numerator and denominator in (28) are identified and, therefore, so is the ratio. Conversely, if the ratio in (28) is identified to equal some value, c_0, then $E_{Q_0}[(\rho(Y^*(t), X) - c_0)\ell(T^*, X)]$ is identified and, by Theorem 4.3, ℓ must belong to the τ-closure of $\mathcal{R}_t$. Hence, under suitable conditions, identification of (28) is equivalent to ℓ belonging to the τ-closure of $\mathcal{R}_t$. These observations complement the independent work of Goff (2025), which derives necessary and sufficient conditions for identification of treatment effects conditional on response type in the discrete instrument models studied by Heckman and Pinto (2018).

4.4.1. *Examples Revisited*

Example 2.4 (cont.) In this context, Corollary 4.4 and the subsequent discussion imply that the expectation of $\rho(Y^*(t), X)\ell(T^*, X)$ is identified if and only if

$$\min_{\{s_j\}_{j=1}^q \subset \mathbf{R}} \sum_{i=1}^r \left(\ell\big(t_i^*, X\big) - \sum_{j=1}^q s_j 1\big\{t_i^*(z_j) = t\big\}\right)^2 = 0 \tag{29}$$

with probability one. If (29) holds, then the expectation of $\rho(Y^*(t), X)\ell(T^*, X)$ equals the expectation of $\rho(Y, X)\kappa(T, Z, X)$ with $\kappa(T, z_j, X) \equiv 1\{T = t\}s_j(X)/P(Z = z_j|X)$ for any $(s_1(X), \dots, s_q(X))$ minimizing (29). These results highlight that identifying a functional about outcomes reduces to a least squares problem when Z is discrete.

Example 2.5 (cont.) In this application, Theorem 4.3 implies that the expectation of $\rho(Y^*(0), X)\ell(K_0^*, K_1^*, X)$ is identified if and only if, for some sequence $\{\nu_j(C, W, X)\}$,

$$\lim_{j\to\infty} E_\mu\left[\left|\ell\left(K_0^*, K_1^*, X\right) - \sum_{c=0}^{1}\int_{K_c^*}^{\bar{\mathbf{w}}} \nu_j(c, w, X)\, dw\right|\right] = 0. \tag{30}$$

Moreover, provided condition (30) holds, the expectation of $\rho(Y^*(0), X)\ell(K_0^*, K_1^*, X)$ is identified as the limit of the expectations of $\rho(Y, X)\kappa_j(T, C, W, X)$ with

$$\kappa_j(t, c, w, X) \equiv \frac{1\{t = 0\}\nu_j(c, w, X)}{f_{W|CX}(w|c, X)P(C = c|X)}.$$

More generally, our analysis yields that the expectation of $\rho(Y^*(t), X)\ell(K_0^*, K_1^*, X)$ is identified if and only if ℓ belongs to the $\|\cdot\|_{\mu,1}$-closure of the set $\mathcal{T}_t$, where

$$\mathcal{T}_0 \equiv \left\{f : f\left(K_0^*, K_1^*, X\right) = g_0\left(K_0^*, X\right) + g_1\left(K_1^*, X\right) \text{ with } g_0(\infty, X) = g_1(\infty, X) = 0\right\},$$
$$\mathcal{T}_1 \equiv \left\{f : f\left(K_0^*, K_1^*, X\right) = g_0\left(K_0^*, X\right) + g_1\left(K_1^*, X\right) \text{ with } g_0(\underline{\mathbf{w}}, X) = g_1(\underline{\mathbf{w}}, X) = 0\right\}.$$

For instance, because μ satisfies $\mu(K_1^* \geq K_0^*) = 1$, it is possible to show that the parameters

$$E_{Q_0}\left[Y^*(1) - Y^*(0)|K_0^* \leq a_0, K_1^* = a_1\right] \quad \text{and} \quad E_{Q_0}\left[Y^*(1) - Y^*(0)|K_0^* = a_0, K_1^* > a_1\right] \tag{31}$$

are identified if and only if the points a_0, a_1 satisfy $\underline{\mathbf{w}} \leq a_1 \leq a_0 < \infty$.

5. ESTIMATION

We have so far allowed features of our model (e.g., μ and ℓ) to depend on P. To construct an estimator, however, we need to specify how these features depend on P. We focus on the leading case in which μ and ℓ are known. The parameter of interest remains

$$\lambda_{Q_0} \equiv E_{Q_0}\left[\ell\left(Y^*, T^*, X\right)\right]. \tag{32}$$

Our estimation strategy is based on two observations that follow from our identification results. First, if the parameter of interest is identified, then it must satisfy

$$\lambda_{Q_0} = \lim_{j\to\infty} E_P\left[\kappa_j(Y, T, Z, X)\right] \tag{33}$$

for some sequence $\{\kappa_j\}$. Second, κ_j can often be set to equal $\kappa_j = \nu_j/\pi$ for some known ν_j and $\pi \equiv dP_{Z|X}/d\mu_{Z|X}$; see, for example, Corollaries 4.1, 4.2, 4.3, and 4.4. The functions $\{\nu_j\}$ are the only feature of our estimation procedure that is model specific.

We follow the literature in deriving a double robust moment condition. To this end, note

$$E_P\left[\frac{f(Z, X)}{\pi(Z, X)}\right] = E_{P_X}\left[E_{\mu_{Z|X}}\left[f(Z, X)\right]\right]$$

for any function f by definition of π. Therefore, the expectation of $\kappa_j = \nu_j/\pi$ satisfies

$$E_P[\kappa_j(Y,T,Z,X)] = E_P\left[\frac{(\nu_j(Y,T,Z,X) - E_P[\nu_j(Y,T,Z,X)|Z,X])}{\pi(Z,X)}\right] + E_{P_X}[E_{\mu_{Z|X}}[E_P[\nu_j(Y,T,Z,X)|Z,X]]]. \quad (34)$$

This moment is double robust in the sense that the equality still holds if we substitute π or the conditional expectation of ν_j with different functions of (Z,X). Double robustness and sample splitting enable estimation through plug-in machine learning methods (Bickel, Klassen, Ritov, and Wellner (1993)). For concreteness, we follow Smucler, Rotnitzky, and Robins (2019), CEIN+ (2022), and Chernozhukov, Newey, and Singh (2022) and employ ℓ_1-regularized estimators in the following algorithm.

STEP 1. Partition $\{1,\dots,n\}$ into K subsets $\{I_k\}_{k=1}^K$. Select functions $\{b_l\}_{l=1}^p$ of (Z,X) with p potentially larger than n, and let $b(Z,X) \equiv (b_1(Z,X),\dots,b_p(Z,X))'$. The number of partitions K is fixed with n, and usually set to five or ten.

STEP 2. For each partition $1 \le k \le K$ and $I_k^c = \{1,\dots,n\} \setminus I_k$, compute the estimators

$$\hat\beta_k \in \arg\min_{\tilde\beta\in\mathbf{R}^p} \sum_{i\in I_k^c} (\nu_j(Y_i,T_i,Z_i,X_i) - b(Z_i,X_i)'\tilde\beta)^2 + \alpha\|\tilde\beta\|_1, \quad (35)$$

$$\hat\gamma_k \in \arg\min_{\tilde\gamma\in\mathbf{R}^p} \sum_{i\in I_k^c} \left(\frac{1}{2}(b(Z_i,X_i)'\tilde\gamma)^2 - E_{\mu_{Z|X}}[b(Z,X_i)'\tilde\gamma]\right) + \alpha\|\tilde\gamma\|_1. \quad (36)$$

The (possibly data-driven) penalty α need not be the same in both optimization problems, but the set of functions $\{b_l\}_{l=1}^p$ should be the same.

STEP 3. For each k, let $|I_k|$ denote the number of observations in I_k and set $\hat\lambda_k$ to equal

$$\hat\lambda_k \equiv \frac{1}{|I_k|}\sum_{i\in I_k}\{b(Z_i,X_i)'\hat\gamma_k(\nu_j(Y_i,T_i,Z_i,X_i) - b(Z_i,X_i)'\hat\beta_k) + E_{\mu_{Z|X}}[b(Z,X_i)'\hat\beta_k]\}.$$

STEP 4. The estimator for λ_{Q_0} is given by $\hat\lambda \equiv \sum_k \hat\lambda_k \times (|I_k|/n)$.

We may view $b(Z,X)'\hat\beta_k$ and $b(Z,X)'\hat\gamma_k$ as estimators for the conditional expectation of ν_j and for $1/\pi$, respectively, and $\hat\lambda$ as a plug-in estimator based on (34). Sample splitting enables us to relax our assumptions, though we note $\hat\lambda$ is asymptotically normally distributed without sample splitting provided we impose sufficiently strong sparsity requirements. We also observe that in step 3 we may substitute $b(Z,X)'\hat\beta_k$ and $b(Z,X)'\hat\gamma_k$ with other suitably convergent machine learning estimators (CCDD+ (2018)).

In order to formally study the asymptotic properties of our estimator, we first define

$$\beta \in \arg\min_{\tilde\beta\in\mathbf{R}^p} E_P[(\nu_j(Y,T,Z,X) - b(Z,X)'\tilde\beta)^2],$$

$$\gamma \in \arg\min_{\tilde\gamma\in\mathbf{R}^p}\left\{\frac{1}{2}E_P[(b(Z,X)'\tilde\gamma)^2] - E_{P_X}[E_{\mu_{Z|X}}[b(Z,X)'\tilde\gamma]]\right\},$$

which are the estimands for which $\hat\beta_k$ and $\hat\gamma_k$ will be assumed to be consistent, and set

$$r_\beta \equiv \max_{1\le k\le K}\{E_P[(b(Z,X)'(\hat\beta_k - \beta))^2]\}^{1/2}, \qquad r_\gamma \equiv \max_{1\le k\le K}\{E_P[(b(Z,X)'(\hat\gamma_k - \gamma))^2]\}^{1/2}.$$

The estimands $b(Z,X)'\beta$ and $b(Z,X)'\gamma$ are approximations to the conditional expectations of ν_j and to $1/\pi$, respectively, and we denote their approximation errors by

$$\delta_\beta \equiv \big\{E_P\big[\big(E_P\big[\nu_j(Y,T,Z,X)|Z,X\big]-b(Z,X)'\beta\big)^2\big]\big\}^{1/2},$$
$$\delta_\gamma \equiv \left\{E_P\left[\left(\frac{1}{\pi(Z,X)}-b(Z,X)'\gamma\right)^2\right]\right\}^{1/2}.$$

Finally, it will be convenient to denote the influence function of our estimator $\hat\lambda$ by

$$\psi(Y,T,Z,X) \equiv b(Z,X)'\gamma\big(\nu_j(Y,T,Z,X)-b(Z,X)'\beta\big)+E_{\mu_{Z|X}}\big[b(Z,X)'\beta\big]-\lambda_{Q_0}, \tag{37}$$

and to let $\sigma^2\equiv \mathrm{Var}_P\{\psi(Y,T,Z,X)\}$. While we have suppressed it from the notation, we note that p (the dimension of $b(Z,X)$) and j (indexing κ_j) can depend on n.

Given the introduced notation, we impose the following assumptions.

ASSUMPTION 5.1: *(i)* $\{Y_i,T_i,X_i,Z_i\}_{i=1}^n$ *is i.i.d.*; *(ii) There are known* $\{\nu_j\}\subseteq L^\infty(P)$ *with* $\kappa_j\equiv\nu_j/\pi$ *satisfying* $\Upsilon(\kappa_j)\to\ell$ *in the* τ *topology*; *(iii)* $\mu_{Z|X}\ll P_{Z|X}$ *and* $\|1/\pi\|_\infty<\infty$.

ASSUMPTION 5.2: *(i)* $\|b'\gamma\|_\infty=O(1)$ *and* $B\equiv\|b'\beta\|_\infty\vee\|\nu_j\|_\infty<\infty$ *satisfies* $B\log(n)=o(\sigma\sqrt{n})$; *(ii)* $r_\beta\vee Br_\gamma\vee\sqrt{n}r_\beta r_\gamma=o_P(\sigma)$; *(iii)* $\sqrt{n}\delta_\beta\delta_\gamma=o(\sigma)$; *(iv)* $\sqrt{n}|\lambda_{Q_0}-E_P[\kappa_j(Y,T,Z,X)]|=o(\sigma)$; *(v)* $|I_k|\asymp n$.

By our identification analysis, Assumption 5.1(ii) is equivalent to identification of λ_{Q_0} in many applications. The condition that ν_j be bounded can be relaxed by strengthening the norm under which $\hat\beta_k$ and $\hat\gamma_k$ converge. The requirements on $\hat\beta_k$ and $\hat\gamma_k$ are stated in Assumption 5.2. We impose high level conditions given the preponderance of results in the literature justifying these assumptions under lower level assumptions; see, for example, Chernozhukov, Newey, and Singh (2022) and Chetverikov and Riis-Vestergaard Sørensen (2025). The rate requirement in Assumption 5.2(iii) is double robust in that it can hold even if δ_β or δ_γ does not tend to zero. Assumption 5.2(iv) automatically holds when $\{\kappa_j\}$ does not depend on j and may be viewed as an undersmoothing requirement otherwise.

Our next result establishes the asymptotic normality of our estimator.

THEOREM 5.1: *Let Assumptions 2.1, 2.2, 4.1, 5.1, 5.2 hold,* λ_{Q_0} *and* ψ *be as defined in (32) and (37), and* $\sigma^2\equiv\mathrm{Var}_P\{\psi(Y,T,Z,X)\}$. *Then there is a* $\mathbb{Z}\sim N(0,1)$ *satisfying*

$$\frac{\sqrt{n}}{\sigma}(\hat\lambda-\lambda_{Q_0})=\frac{1}{\sqrt{n}\sigma}\sum_{i=1}^n\psi(Y_i,T_i,Z_i,X_i)+o_P(1)=\mathbb{Z}+o_P(1). \tag{38}$$

For inference, we rely on a multiplier bootstrap procedure that readily extends to vector valued parameters and their functionals. Specifically, for each k we set

$$\hat\psi_k(Y,T,Z,X) \equiv b(Z,X)'\hat\gamma_k\big(\nu_j(Y,T,Z,X)-b(Z,X)'\hat\beta_k\big)+E_{\mu_{Z|X}}\big[b(Z,X)'\hat\beta_k\big]-\hat\lambda. \tag{39}$$

Our bootstrapped estimator $\hat{\lambda}^*$ is then obtained by employing $\hat{\psi}_k$ and an i.i.d. sample $\{W_i\}_{i=1}^n$ of standard normal weights independent of the data to perturb $\hat{\lambda}$ according to

$$\hat{\lambda}^* \equiv \hat{\lambda} + \frac{1}{n}\sum_{k=1}^{K}\sum_{i\in I_k} W_i\hat{\psi}_k(Y_i, T_i, Z_i, X_i).$$

We employ Gaussian weights for simplicity, though under appropriate moment restrictions the proposed bootstrap remains valid provided W satisfies $E[W]=0$ and $E[W^2]=1$.

The next result establishes the validity of the proposed bootstrap.

THEOREM 5.2: *Let the conditions of Theorem 5.1 hold and $\{W_i\}_{i=1}^n$ be i.i.d. standard normal random variables independent of $\{Y_i, T_i, Z_i, X_i\}_{i=1}^n$. Then there exists a standard normal random variable $\mathbb{Z}^*$ independent of $\{Y_i, T_i, Z_i, X_i\}_{i=1}^n$ and satisfying*

$$\frac{\sqrt{n}}{\sigma}(\hat{\lambda}^* - \hat{\lambda}) = \frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n} W_i\psi(Y_i, T_i, Z_i, X_i) + o_P(1) = \mathbb{Z}^* + o_P(1). \tag{40}$$

Theorems 5.1, 5.2, and the Delta method justify employing the bootstrap to conduct inference on, for example, conditional expectations of potential outcomes given types or of types given covariates. These parameters equal $F(\lambda_{1Q_0}, \dots, \lambda_{qQ_0})$ for some known differentiable function F and each λ_{iQ_0} with the structure in (32). For instance, to obtain a two-sided confidence region we: (i) Compute estimators $(\hat{\lambda}_1, \dots, \hat{\lambda}_q)$ for $(\lambda_{1Q_0}, \dots, \lambda_{qQ_0})$; (ii) Draw B samples $\{W_i^{(b)}\}_{i=1}^n$ independent of the data; (iii) Employ each $\{W_i^{(b)}\}_{i=1}^n$ to obtain bootstrap estimators $(\hat{\lambda}_1^{*(b)}, \dots, \hat{\lambda}_q^{*(b)})$; (iv) Set $\hat{c}_\alpha$ to equal the $1-\alpha$ quantile of $\{|F(\hat{\lambda}_1^{*(b)}, \dots, \hat{\lambda}_q^{*(b)}) - F(\hat{\lambda}_1, \dots, \hat{\lambda}_q)|\}_{b=1}^B$; and (v) Report $F(\hat{\lambda}_1, \dots, \hat{\lambda}_q) \pm \hat{c}_\alpha$ as a two-sided confidence region. Our results also allow us to conduct inference when F is directionally differentiable by using the framework in Fang and Santos (2018).

We conclude by illustrating our results in the context of Examples 2.4 and 2.5. For calculations, it is helpful to note that the estimator is invariant to scaling μ and it is therefore not necessary to normalize μ to be a probability measure.

Example 2.4 (cont.) Recall that in this example Z had discrete support $\mathbf{Z} = \{z_1, \dots, z_q\}$ and the support of T^* was restricted to $\mathbf{R}^* \equiv \{t_1^*, \dots, t_r^*\}$. We consider estimation of

$$E_{Q_0}\big[\rho\big(Y^*(t)\big)\big|T^* \in A\big] \tag{41}$$

for some $A \subseteq \mathbf{R}^*$. In (29), we showed $E_{Q_0}[\rho(Y^*(t))1\{T^* \in A\}]$ is identified if and only if

$$\min_{\{s_j\}_{j=1}^q \subset \mathbf{R}} \sum_{i=1}^{r}\left(1\{t_i^* \in A\} - \sum_{j=1}^{q} s_j 1\{t_i^*(z_j) = t\}\right)^2 = 0. \tag{42}$$

For any minimizer $\{s_j\}$ to (42), we may then estimate $E_{Q_0}[\rho(Y^*(t))1\{T^* \in A\}]$ by setting

$$\nu(Y, T, Z, X) = \rho(Y)1\{T = t\}\left(\sum_{j=1}^{q} s_j 1\{Z = z_j\}\right), \qquad E_{\mu_{Z|X}}\big[b(Z, X_i)\big] = \sum_{j=1}^{q} b(z_j, X_i)$$

in (35) and (36). Letting $\rho(Y) = 1$, we obtain an estimator for $E_{Q_0}[1\{T^* \in A\}]$. We estimate (41) by dividing the estimator for $E_{Q_0}[\rho(Y^*(t))1\{T^* \in A\}]$ by the estimator for

$E_{Q_0}[1\{T^* \in A\}]$. In this example, Assumptions 5.1(ii)(iii) holds provided ρ is bounded and $P(Z = z|X)$ is bounded away from zero for all $z \in \mathbf{Z}$, while Assumption 5.2(iv) is automatic because ν does not depend on j and, therefore, neither does $\kappa = \nu/\pi$.

Example 2.5 (cont.) Recall T^* is represented by (K_0^*, K_1^*) (see (24)). We illustrate our estimator in the context of estimating the expectation of a function ℓ of (K_0^*, X). First, suppose $\ell(K_0^*, X)$ is differentiable in K_0^* on $(\underline{\mathbf{w}}, \bar{\mathbf{w}})$ with derivative $\ell'(K_0^*, X)$ and $\ell(K_0^*, X) = 0$ when $K_0^* \in \{\underline{\mathbf{w}}, \bar{\mathbf{w}}, +\infty\}$. Result (25) implies $E_{Q_0}[\ell(K_0^*, X)]$ can be estimated by setting

$$\nu(Y, T, C, W, X) = 1\{T = 1, C = 0\}\ell'(W, X), \tag{43}$$

$$E_{\mu_{Z|X}}\big[b(C, W, X_i)\big] = \sum_{c=0}^{1} \int_{\underline{\mathbf{w}}}^{\bar{\mathbf{w}}} b(c, w, X_i)\, dw \tag{44}$$

in (35) and (36). Expectations of other functions of (K_0^*, X) can be estimated by approximating them with differentiable functions. For example, the expectation of $\ell(K_0^*) = 1\{a \le K_0^* \le b\}$ with $\underline{\mathbf{w}} < a < b < \bar{\mathbf{w}}$ can be approximated by the expectation of $\ell_j(K_0^*) = F((b - K_0^*)/h_j) - F((a - K_0^*)/h_j)$ for some $h_j \downarrow 0$ and F the c.d.f. of a compactly supported mean zero continuous random variable. In this case, (43) becomes

$$\nu_j(Y, T, C, W, X) = 1\{T = 1, C = 0\} \times \frac{1}{h_j}\left(F'\left(\frac{a - W}{h_j}\right) - F'\left(\frac{b - W}{h_j}\right)\right). \tag{45}$$

Assumptions 5.1(ii)(iii) require $h_j = o(1)$ and $f_{W|C}(W|c, X)P(C = c|X)$ be bounded away from zero. Under regularity conditions, $|\lambda_{Q_0} - E_P[\kappa_j(T, Z, X)]| = O(h^2)$ and $B \asymp \sigma^2 \asymp 1/h_j$ so Assumptions 5.2(i)(iv) hold when $\log^2(n)/(nh_j) = o(1)$ and $nh_j^5 = o(1)$.

APPENDIX A: PROOFS FOR SECTION 4

Throughout, we let Q_V denote the marginal distribution of a random variable V under Q and $Q_{V|W}$ the conditional distribution of V given W. Distributions Q for (Y^*, T^*, Z, X) are defined on a product σ-field generated by $\mathcal{F}_{Y^*} \times \mathcal{F}_{T^*} \times \mathcal{F}_Z \times \mathcal{F}_X$, where $\mathcal{F}_V$ denotes the σ-field on which Q_V is defined. In the main text we implicitly assumed $P_{Z|X}$, $\bar{Q}_{Y^*|X}$, and $\bar{Q}_{Y^*(t)|X}$ are well defined,. Since $Y^*(t) \in \mathbf{R}$, these conditional distributions exist provided $\mathbf{X}$ and $\mathbf{Z}$ are Souslin spaces (e.g., Euclidean spaces) and $\mathcal{F}_Z$ and $\mathcal{F}_X$ are Borel σ-algebras; see Corollary 10.4.6 in Bogachev (2007). For any (V_1, V_2, W) with distribution λ, we understand $V_1 \perp\!\!\!\perp V_2|W$ under λ to mean that, for $i \ne j \in \{1, 2\}$ and any bounded f

$$E_\lambda\big[f(V_j, W)|V_i, W\big] = E_\lambda\big[f(V_j, W)|W\big]. \tag{46}$$

We note that if (46) holds for $(i, j) = (1, 2)$, then it holds for $(j, i) = (2, 1)$.

PROOF OF LEMMA 4.1: Lemma 13.14 and Corollary 3.5 in Aliprantis and Border (2006) imply $D_0 \equiv \{dQ/d\mu : Q \in \Theta_0\} \subset L^1(\mu)$ is separable under $\|\cdot\|_{\mu,1}$. Therefore, there is a countable $\mathcal{D} \equiv \{Q_i\}_{i=1}^\infty \subseteq \Theta_0$ such that $\{dQ_i/d\mu\}_{i=1}^\infty$ is dense in D_0. Next, for any $2 \le n < \infty$ let

$$\lambda_{in} = \begin{cases} 1 - \sum_{i=2}^{n} \lambda_{in} & \text{if } i = 1, \\ 2^{-i}/\max\big\{1, \|dQ_i/d\mu\|_{\mathbf{Q}}\big\} & \text{if } 2 \le i \le n, \\ 0 & \text{if } i > n, \end{cases} \tag{47}$$

noting that $\sum_{i=1}^n \lambda_{in} = 1$ and $\lambda_{in} > 0$ for any $1 \le i \le n$ due to $\sum_{i=1}^\infty 2^{-i} = 1$. Moreover, $f_n \equiv \sum_{i=1}^n \lambda_{in}(dQ_i/d\mu)$ belongs to $\mathcal{Q}$ for all n by convexity of $\mathcal{Q}$ and for any $n < m$,

$$\|f_n - f_m\|_{\mathbf{Q}} \le |\lambda_{1n} - \lambda_{1m}| \left\| \frac{dQ_1}{d\mu} \right\|_{\mathbf{Q}} + \sum_{i=n+1}^{\infty} \frac{1}{2^i}. \tag{48}$$

Since $1/2 < \lambda_{1n}$ and λ_{1n} is decreasing in n, the sequence $\{\lambda_{1n}\}_{n=1}^\infty$ has a limit in $\mathbf{R}$. Hence, by (48), $\{f_n\}_{n=1}^\infty$ is Cauchy in $\mathbf{Q}$ and, since $\mathbf{Q}$ is complete and $\mathcal{Q} \subseteq \mathbf{Q}$ is closed, $\|f_n - q_0\|_{\mathbf{Q}} = o(1)$ for some $q_0 \in \mathcal{Q} \subseteq L^1(\mu)$. Finally, we define $\bar{Q}$ by setting $\bar{Q}(A) \equiv \int_A q_0 \, d\mu$. Since $\|\cdot\|_{\mu,1} \lesssim \|\cdot\|_{\mathbf{Q}}$, we have $\|f_n - q_0\|_{\mu,1} = o(1)$ and, therefore,

$$\lim_{n\to\infty} \left| \int g \, d\bar{Q} - \sum_{i=1}^n \lambda_{in} \int g \, dQ_i \right| \le \lim_{n\to\infty} \|g\|_\infty \int \left| q_0 - \sum_{i=1}^n \lambda_{in} \frac{dQ_i}{d\mu} \right| d\mu = 0 \tag{49}$$

for any bounded g. In particular, result (49) implies that $\int d\bar{Q} = 1$ and $0 \le \bar{Q}(A) \le 1$.

We next show $\bar{Q} \in \Theta_0$. Note (49) and $Q_i \in \Theta_0$ imply, for any $t \in \mathbf{T}$ and (measurable) V,

$$\begin{aligned}
\bar{Q}\big(T^*(Z) = t, \big(Y^*(t), Z, X\big) \in V\big) &= \lim_{n\to\infty} \sum_{i=1}^n \lambda_{in} Q_i\big(T^*(Z) = t, \big(Y^*(t), Z, X\big) \in V\big) \\
&= \lim_{n\to\infty} \sum_{i=1}^n \lambda_{in} P\big(T = t, (Y, Z, X) \in V\big) \\
&= P\big(T = t, (Y, Z, X) \in V\big),
\end{aligned} \tag{50}$$

since $\sum_{i=1}^n \lambda_{in} = 1$, which implies $\bar{Q}$ induces P. Next, note that, for any bounded f and g,

$$\begin{aligned}
&E_{\bar{Q}}\big[g(Z, X) f\big(Y^*, T^*, X\big)\big] \\
&\quad = \lim_{n\to\infty} \sum_{i=1}^n \lambda_{in} E_{Q_i}\big[g(Z, X) E_{Q_i}\big[f\big(Y^*, T^*, X\big) | X\big]\big] \\
&\quad = E_{\bar{Q}}\left[g(Z, X) \left(\lim_{n\to\infty} \sum_{i=1}^n \lambda_{in} E_{Q_i}\big[f\big(Y^*, T^*, X\big) | X\big] \right) \right],
\end{aligned} \tag{51}$$

where the first equality follows from (49) and $(Y^*, T^*) \perp\!\!\!\perp Z | X$ under Q_i, and the second from $Q_i \in \Theta_0$, $\bar{Q}$ inducing P by (50), and the dominated convergence theorem. Since (51) holds for any bounded g, Definition 10.1.1 in Bogachev (2007) yields

$$\begin{aligned}
E_{\bar{Q}}\big[f\big(Y^*, T^*, X\big) | X, Z\big] &= \lim_{n\to\infty} \sum_{i=1}^n \lambda_{in} E_{Q_i}\big[f\big(Y^*, T^*, X\big) | X\big] \\
&= E_{\bar{Q}}\big[f\big(Y^*, T^*, X\big) | X\big],
\end{aligned} \tag{52}$$

where the second equality follows by applying the equalities in (51) evaluated at functions g of X only. Therefore, $(Y^*, T^*) \perp\!\!\!\perp Z | X$ under $\bar{Q}$ and, by the preceding results, we conclude $\bar{Q} \in \Theta_0$. Next, fix an arbitrary $Q \in \Theta_0$ and a set A with $Q(A) > 0$, and note $\|dQ/d\mu - dQ_k/d\mu\|_{\mu,1} < Q(A)/2$ for some k because $\{dQ_i/d\mu\}_{i=1}^{\infty}$ is dense in D_0. Hence,

$$Q_k(A) \geq Q(A) - \left|\int_A \left(\frac{dQ}{d\mu} - \frac{dQ_k}{d\mu}\right) d\mu\right| \geq Q(A) - \left\|\frac{dQ}{d\mu} - \frac{dQ_k}{d\mu}\right\|_{\mu,1} > \frac{Q(A)}{2} \tag{53}$$

and, therefore, $Q_k(A) > 0$. Since $\{\lambda_{kn}\}_{n=1}^{\infty}$ is bounded away from zero for n sufficiently large, results (49) and (53) imply $\bar{Q}(A) > 0$. Since A was arbitrary, it follows that $Q \ll \bar{Q}$.

To establish Θ_0 is convex, let $Q_1, Q_2 \in \Theta_0$, $\gamma \in [0, 1]$, and define $Q_\gamma \equiv \gamma Q_1 + (1-\gamma)Q_2$. Then note: (i) $dQ_\gamma/d\mu = \gamma\, dQ_1/d\mu + (1-\gamma)\, dQ_2/d\mu \in \mathcal{Q}$ by Assumption 2.2(iii); (ii) Q_γ induces P by the arguments in (50) applied with Q_γ in place of $\bar{Q}$; and (iii) $(Y^*, T^*) \perp\!\!\!\perp Z|X$ under Q_γ by the arguments in (51) and (52) applied with Q_γ in place of $\bar{Q}$. *Q.E.D.*

PROOF OF LEMMA 4.2: First, note $\bar{Q}(\Upsilon(\kappa) = \ell) = 1$ and Lemma 4.1 imply $Q(\Upsilon(\kappa) = \ell) = 1$ for all $Q \in \Theta_0$. Therefore, $\ell(Y^*, T^*, X) = E_Q[\kappa(Y, T, Z, X)|Y^*, T^*, X]$ for all $Q \in \Theta_0$ by Corollary A.1, which implies $\ell \in L^1(Q)$ by Jensen's inequality and $\kappa \in L^1(P)$. Moreover, for any $Q \in \Theta_0$ we obtain $E_Q[\ell(Y^*, T^*, X)] = E_Q[\kappa(Y, T, Z, X)] = E_P[\kappa(Y, T, Z, X)]$ by the law of iterated expectations, and hence the lemma follows. *Q.E.D.*

PROOF OF COROLLARY 4.1: By Lemma A.2, we have $\mu(\Upsilon(\kappa) = \ell) = 1$ and hence also that $\bar{Q}(\Upsilon(\kappa) = \ell) = 1$ due to $\bar{Q} \ll \mu$. Thus, the result is immediate from Lemma 4.2. *Q.E.D.*

PROOF OF THEOREM 4.1: We first show λ_{Q_0} being identified implies ℓ belongs to the τ-closure of $\mathcal{R}$. Let $\mathcal{B}'$ denote the linear span of $\mathcal{R} \cup \{\ell\}$ and note every $B \in \mathcal{B}$ is a linear functional on $\mathcal{B}'$. We endow $\mathcal{B}'$ with the weak topology generated by $\mathcal{B}$, denoted $\sigma(\mathcal{B}', \mathcal{B})$, and note $(\mathcal{B}', \sigma(\mathcal{B}', \mathcal{B}))$ is a topological vector space whose dual is $\mathcal{B}$. Moreover, Theorem 5.73 in Aliprantis and Border (2006) implies $(\mathcal{B}', \sigma(\mathcal{B}', \mathcal{B}))$ is locally convex, while Lemma 2.53 in Aliprantis and Border (2006) implies the τ topology on $\mathcal{R} \cup \{\ell\}$ equals the relative topology on $\mathcal{R} \cup \{\ell\}$ induced by the topology $\sigma(\mathcal{B}', \mathcal{B})$ on $\mathcal{B}'$. Therefore, ℓ belongs to the τ-closure of $\mathcal{R}$ (in $\mathcal{R} \cup \{\ell\}$) if and only if ℓ belongs to the $\sigma(\mathcal{B}', \mathcal{B})$-closure of $\mathcal{R}$ in $\mathcal{B}'$, denoted $\bar{\mathcal{R}}$. We show ℓ belongs to the τ-closure of $\mathcal{R}$ by showing $\ell \in \bar{\mathcal{R}}$.

We proceed by contradiction and suppose $\ell \notin \bar{\mathcal{R}}$. Since $\mathcal{B}$ is the dual of $(\mathcal{B}', \sigma(\mathcal{B}', \mathcal{B}))$, Corollary 5.80 in Aliprantis and Border (2006) implies there is $B_0 \in \mathcal{B}$ satisfying $B_0(\ell) \neq 0$ and $B_0(B') = 0$ for all $B' \in \bar{\mathcal{R}}$. Moreover, by definition of $\mathcal{B}$ there is $\{(s_j, Q_j)\}_{j=1}^{J}$ with $s_j \in \mathcal{S}_{Q_j}$ and $Q_j \in \Theta_0$ for all $1 \leq j \leq J < \infty$, and $B_0(B') = B'(\sum_{j=1}^{J} \langle \cdot, s_j \rangle_{Q_j})$ for all $B' \in \mathcal{B}'$. Hence, $B_0(B') = 0$ for all $B' \in \bar{\mathcal{R}}$ implies, for any $f \in L^1(P)$ that

$$\begin{aligned} 0 &= \sum_{j=1}^{J} E_{Q_j}\left[\left(\sum_{t \in \mathbf{T}} E_{P_{Z|X}}\left[f\left(Y^*(t), t, Z, X\right) 1\left\{T^*(Z) = t\right\}\right]\right) s_j\left(Y^*, T^*, X\right)\right] \\ &= E_P\left[f(Y, T, Z, X)\left(\sum_{j=1}^{J} E_{Q_j}\left[s_j\left(Y^*, T^*, X\right)|Y, T, Z, X\right]\right)\right], \end{aligned} \tag{54}$$

where the second equality follows from Corollary A.1, the law of iterated expectations, and $Q_j \in \Theta_0$. It follows $P(\sum_{j=1}^{J} E_{Q_j}[s_j(Y^*, T^*, X)|Y, T, Z, X] = 0) = 1$ because (54) holds for all $f \in L^1(P)$. Therefore, by Lemma A.3 there are $\tilde{Q}, Q^{\mathrm{a}} \in \Theta_0$, and $\eta > 0$ such that $\tilde{Q}(A) = Q^{\mathrm{a}}(A) + \eta \sum_{j=1}^{J} E_{Q_j}[s_j(Y^*, T^*, X)1\{(Y^*, T^*, Z, X) \in A\}]$ for all A. Letting $\mathbf{1}$ denote the constant function taking value one, we then obtain

$$\lambda_{\tilde{Q}} = \langle \ell, \mathbf{1}\rangle_{Q^{\mathrm{a}}} + \eta \sum_{j=1}^{J} \langle \ell, s_j\rangle_{Q_j} = \lambda_{Q^{\mathrm{a}}} + \eta B_0(\ell).$$

However, $\eta > 0$ and $B_0(\ell) \neq 0$ together imply that $\lambda_{\tilde{Q}} \neq \lambda_{Q^{\mathrm{a}}}$. Thus, since $\tilde{Q}, Q^{\mathrm{a}} \in \Theta_0$ we obtain that λ_{Q_0} is not identified, reaching a contradiction.

For the converse, suppose ℓ belongs to the τ-closure of $\mathcal{R}$. Since $\mathcal{B}$ is identified (because it only depends on Θ_0), ℓ is identified, and $\mathcal{R}$ is identified (because $\Upsilon : L^1(P) \to L^1(\bar{Q})$ is identified), Theorem 2.4 in Aliprantis and Border (2006) implies there is an identified net $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{R}$ satisfying $\lim_\alpha B(f_\alpha) = B(\ell)$ for all $B \in \mathcal{B}$. Hence, for any $Q_1, Q_2 \in \Theta_0$

$$\lambda_{Q_1} = \langle \ell, \mathbf{1}\rangle_{Q_1} = \lim_{\alpha} \langle f_\alpha, \mathbf{1}\rangle_{Q_1} = \lim_{\alpha} \langle f_\alpha, \mathbf{1}\rangle_{Q_2} = \langle \ell, \mathbf{1}\rangle_{Q_2} = \lambda_{Q_2},$$

where the first and last equalities follow by definition of λ_Q, the second and fourth equalities from $\lim_\alpha B(f_\alpha) = B(\ell)$ for all $B \in \mathcal{B}$, and the third equality by Lemma 4.2 and $f_\alpha \in \mathcal{R}$. Thus, we conclude that λ_Q is constant in $Q \in \Theta_0$ and that λ_{Q_0} is identified. *Q.E.D.*

PROOF OF COROLLARY 4.2: To apply Theorem 4.1 note Assumptions 2.1 and 2.2 are imposed. Moreover, Assumption 4.1(i) holds due to $\ell \in L^1(\mu)$, $dQ/d\mu \in L^\infty(\mu)$ for all $Q \in \Theta_0$ by Assumption 2.2(iii), and Holder's inequality implying $\int |\ell|\, dQ \leq \|\ell\|_{\mu,1} \|\frac{dQ}{d\mu}\|_{\mu,\infty} < \infty$ for any $Q \in \Theta_0$. Assumption 4.1(ii) is immediate because $\mathcal{Q} = \mathbf{Q}$. Thus, Theorem 4.1 implies that λ_Q is identified if and only if ℓ belongs to the τ-closure of $\mathcal{R}$.

To establish part (i), note $\ell \in L^1(Q)$ for all $Q \in \Theta_0$ and $s\, d\bar{Q}/d\mu \in L^\infty(\mu)$ for any $s \in L^\infty(\bar{Q}_{Y^*T^*X})$ (because $\bar{Q} \ll \mu$ and $d\bar{Q}/d\mu \in L^\infty(\mu)$) imply $\mathcal{S}_{\bar{Q}} = L^\infty(\bar{Q}_{Y^*T^*X})$. Therefore, if ℓ belongs to the τ-closure of $\mathcal{R}$, then $\|\ell - \Upsilon(\kappa_j)\|_{\bar{Q},1} = o(1)$ for some $\{\kappa_j\} \subseteq L^1(P)$ by Lemma A.4. Since $\mu \ll \bar{Q}$ and $d\mu/d\bar{Q}$ is bounded, $\{\kappa_j\}$ satisfies $\|\ell - \Upsilon(\kappa_j)\|_{\mu,1} = o(1)$. For the converse, note that if $\|\ell - \Upsilon(\kappa_j)\|_{\mu,1} = o(1)$, then $\|\ell - \Upsilon(\kappa_j)\|_{Q,1} = o(1)$ for all $Q \in \Theta_0$ due to Holder's inequality and $dQ/d\mu \in L^\infty(\mu)$. Hence, for any $Q \in \Theta_0$, $\int \ell\, dQ = \int \Upsilon(\kappa_j)\, dQ + o(1)$ and Lemma 4.2 implies $\lambda_Q = E_P[\kappa_j(Y, T, Z, X)] + o(1)$. Part (ii) follows from part (i) and Lemma A.2. *Q.E.D.*

PROOF OF THEOREM 4.2: By Theorem 4.1, λ_{Q_0} is identified if and only if ℓ belongs to the τ-closure of $\mathcal{R}$. Since $\mathcal{R}_T \subseteq \mathcal{R}$, λ_{Q_0} is identified when ℓ is in the τ-closure of $\mathcal{R}_T$. We show the converse by showing that if ℓ is in the τ-closure of $\mathcal{R}$, then it belongs to the τ-closure of $\mathcal{R}_T$. If ℓ belongs to the τ-closure of $\mathcal{R}$, then Theorem 2.14 in Aliprantis and Border (2006) implies that there is a net $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subseteq L^1(P)$ satisfying

$$\lim_{\alpha} \langle s, \Upsilon(f_\alpha)\rangle_Q = \langle s, \ell\rangle_Q \tag{55}$$

for all $s \in \mathcal{S}_Q$, $Q \in \Theta_0$. Set $g_\alpha(t, Z, X) \equiv E_{\bar{Q}_{Y^*|X}}[f_\alpha(Y^*(t), t, Z, X)]$ for any $t \in \mathbf{T}$. By Jensen's inequality, $\bar{Q} \in \Theta_0$, Lemma A.1, $\bar{Q}^{\mathrm{it}}_{ZX} = \bar{Q}_{ZX}$, and $\bar{Q}^{\mathrm{it}}_{T^*X} = \bar{Q}_{T^*X}$ we have

$$E_P\big[|g_\alpha(T, Z, X)|\big]$$

$$
\begin{aligned}
&= E_{\bar{Q}_{T^*ZX}}\Bigg[\sum_{t\in\mathbf{T}} 1\{T^*(Z)=t\}\big|E_{\bar{Q}_{Y^*|X}}\big[f_\alpha\big(Y^*(t),t,Z,X\big)\big]\big|\Bigg] \\
&\le E_{\bar{Q}^{\mathrm{it}}_{Y^*T^*X}}\Bigg[\sum_{t\in\mathbf{T}} E_{\bar{Q}_{Z|X}}\big[1\{T^*(Z)=t\}\big|f_\alpha\big(Y^*(t),t,Z,X\big)\big|\big]\Bigg]. \qquad (56)
\end{aligned}
$$

Since the constant function taking value one belongs to $\mathcal{S}_{\bar{Q}}$, Assumption 4.2(ii) implies that $d\bar{Q}^{\mathrm{it}}_{Y^*T^*X}/d\bar{Q}_{Y^*T^*X} \in \mathcal{S}_{\bar{Q}} \subseteq L^\infty(\bar{Q}_{Y^*T^*X})$. Therefore, $d\bar{Q}^{\mathrm{it}}_{Y^*T^*X}/d\bar{Q}_{Y^*T^*X}$ is bounded, which together with (56), Corollary A.1, and $\bar{Q}\in\Theta_0$ imply $E_P[|g_\alpha(T,Z,X)|] \lesssim E_P[|f_\alpha(Y,T,Z,X)|]$. Thus, since $f_\alpha\in L^1(P)$, it follows $g_\alpha \in L^1(P_{TZX})$.

For any $s_0\in\mathcal{S}_{\bar{Q}}\cap L^\infty(\bar{Q}_{T^*X})$, note $\bar{Q}^{\mathrm{it}}_{Z|X}=\bar{Q}_{Z|X}=P_{Z|X}$ (because $\bar{Q}\in\Theta_0$) implies

$$
\begin{aligned}
\langle s_0, \Upsilon(g_\alpha)\rangle_{\bar{Q}} &= E_{\bar{Q}_{T^*X}}\Bigg[E_{P_{Z|X}}\Bigg[E_{\bar{Q}_{Y^*|X}}\Bigg[\sum_{t\in\mathbf{T}} 1\{T^*(Z)=t\}f_\alpha\big(Y^*(t),t,Z,X\big)s_0\big(T^*,X\big)\Bigg]\Bigg]\Bigg] \\
&= E_{\bar{Q}^{\mathrm{it}}_{Y^*T^*X}}\Bigg[\sum_{t\in\mathbf{T}} E_{\bar{Q}^{\mathrm{it}}_{Z|X}}\big[1\{T^*(Z)=t\}f_\alpha\big(Y^*(t),t,Z,X\big)\big]s_0\big(T^*,X\big)\Bigg]. \qquad (57)
\end{aligned}
$$

Moreover, since s_0 and ℓ only depend on (T^*,X), it follows from $\bar{Q}_{T^*X}=\bar{Q}^{\mathrm{it}}_{T^*X}$ that

$$
\begin{aligned}
\langle \ell, s_0\rangle_{\bar{Q}} &= \langle \ell, s_0\rangle_{\bar{Q}_{T^*X}} = \langle \ell, s_0\rangle_{\bar{Q}^{\mathrm{it}}_{T^*X}} = \left\langle \ell, s_0\frac{d\bar{Q}^{\mathrm{it}}_{Y^*T^*X}}{d\bar{Q}_{Y^*T^*X}}\right\rangle_{\bar{Q}} \\
&= \lim_\alpha\left\langle \Upsilon(f_\alpha), s_0\frac{d\bar{Q}^{\mathrm{it}}_{Y^*T^*X}}{d\bar{Q}_{Y^*T^*X}}\right\rangle_{\bar{Q}} = \lim_\alpha\langle \Upsilon(f_\alpha), s_0\rangle_{\bar{Q}^{\mathrm{it}}} = \lim_\alpha\langle \Upsilon(g_\alpha), s_0\rangle_{\bar{Q}}, \qquad (58)
\end{aligned}
$$

where the fourth equality follows from (55), $\bar{Q}\in\Theta_0$, and $s_0(d\bar{Q}^{\mathrm{it}}_{Y^*T^*X}/d\bar{Q}_{Y^*T^*X})\in\mathcal{S}_{\bar{Q}}$ by Assumption 4.2(ii), and the sixth from result (57) and $\bar{Q}^{\mathrm{it}}_{Z|X}=\bar{Q}_{Z|X}=P_{Z|X}$. Next, set $\Pi_Q(s)(T^*,X)\equiv E_Q[s(Y^*,T^*,X)|T^*,X]$ and note for any $s\in\mathcal{S}_Q$ and $Q\in\Theta_0$ we have

$$
\langle \ell, s\rangle_Q = \left\langle \ell, \Pi_Q(s)\frac{dQ_{T^*X}}{d\bar{Q}_{T^*X}}\right\rangle_{\bar{Q}} = \lim_\alpha\left\langle \Upsilon(g_\alpha), \Pi_Q(s)\frac{dQ_{T^*X}}{d\bar{Q}_{T^*X}}\right\rangle_{\bar{Q}} = \lim_\alpha\langle\Upsilon(g_\alpha), s\rangle_Q, \qquad (59)
$$

where the first equality follows from ℓ depending only on (T^*,X), and the second from (58) and $\Pi_Q(s)\,dQ_{T^*X}/d\bar{Q}_{T^*X}\in\mathcal{S}_{\bar{Q}}$ by Assumptions 4.2(iii)(iv). Result (59) and Theorem 2.14 in Aliprantis and Border (2006) then imply ℓ belongs to the τ-closure of $\mathcal{R}_T$. *Q.E.D.*

PROOF OF COROLLARY 4.3: If $\bar{Q}(\Upsilon(\kappa)=\ell)=1$ for some $\kappa\in L^1(P_{TZX})$, then λ_{Q_0} is identified by Lemma 4.2. To establish the converse, we verify the conditions for Theorem 4.2. Note Assumptions 2.1 and 2.2 are assumed, and Assumption 4.1(ii) holds because $\mathcal{Q}=\mathbf{Q}=L^1(\mu)$. Define $\mathbf{T}^*_0\equiv\{t^*\in\mathbf{T}^*:\bar{Q}(T^*=t^*)>0\}$ and note that $Q(T^*\in\mathbf{T}^*_0)=1$ for any $Q\in\Theta_0$ due to $Q\ll\bar{Q}$. Since $Q_X=\bar{Q}_X=P_X$ for any $Q\in\Theta_0$, it follows that

$$
\frac{dQ_{T^*X}}{d\bar{Q}_{T^*X}}\big(T^*,X\big) = \sum_{t^*\in\mathbf{T}^*_0} 1\{T^*=t^*\}\frac{Q\big(T^*=t^*|X\big)}{\bar{Q}\big(T^*=t^*|X\big)} \le \frac{1}{\varepsilon} \qquad (60)
$$

because $\bar{Q}(T^* = t^*|X) \geq \varepsilon$ almost surely. Hence, $\ell \in L^1(\bar{Q}_{T^*X})$ and (60) imply $\ell \in L^1(Q)$ for any $Q \in \Theta_0$, verifying Assumption 4.1(i). Moreover, since $\mathcal{S}_Q = L^\infty(Q_{Y^*T^*X})$ for any $Q \in \Theta_0$ and $d\bar{Q}^{\mathrm{it}}/d\bar{Q}$ is bounded, it follows from (60) that Assumption 4.2 also holds. Thus, we may apply Theorem 4.2 together with Lemma A.4 to conclude that if λ_{Q_0} is identified, then there exists $\{\kappa_j\}_{j=1}^\infty \subseteq L^1(P_{TZX})$ satisfying $\|\ell - \Upsilon(\kappa_j)\|_{\bar{Q},1} = o(1)$.

Next, let $\mathbf{S}_0 \equiv \{(t, z) \in \mathbf{T} \times \mathbf{Z} : P(T = t, Z = z) > 0\}$ and note that for any $(t, z) \in \mathbf{S}_0$

$$P(T = t, Z = z|X) = \sum_{t^* \in \mathbf{T}_0^*: t^*(z) = t} \bar{Q}(T^* = t^*|X)P(Z = z|X) \geq \varepsilon^2 \tag{61}$$

because $\bar{Q} \in \Theta_0$, $(Y^*, T^*) \perp\!\!\!\perp Z|X$, $\bar{Q}(T^* = t^*|X) \geq \varepsilon$ for any $t^* \in \mathbf{T}_0^*$, and $P(Z = z|X) \geq \varepsilon$ for any $z \in \mathbf{Z}$. Hence, for any event E with $P(X \in E) > 0$ and $(t, z) \in \mathbf{S}_0$,

$$P(X \in E) = \frac{P(X \in E|T = t, Z = z)P(T = t, Z = z)}{P(T = t, Z = z|X \in E)} \leq \frac{P(X \in E|T = t, Z = z)}{\varepsilon^2}. \tag{62}$$

Let $P_{X|t,z}$ denote the distribution of X conditional on $(T, Z) = (t, z)$ for any $(t, z) \in \mathbf{S}_0$. Then note (62) implies $dP_X/dP_{X|t,z} \leq \varepsilon^{-2}$, which together with (61) yields $\kappa_j(t, z, \cdot) \in L^1(P_X)$ for any $(t, z) \in \mathbf{S}_0$. Next, set $\tilde{\kappa}_j(t, z, X) \equiv \kappa_j(t, z, X)P(Z = z|X)$ and note

$$\|\ell - \Upsilon(\kappa_j)\|_{\bar{Q},1} = E_{\bar{Q}}\left[\left|\ell(T^*, X) - \sum_{(t,z) \in \mathbf{S}_0} 1\{T^*(z) = t\}\tilde{\kappa}_j(t, z, X)\right|\right]. \tag{63}$$

Since $\tilde{\kappa}_j(t, z, \cdot) \in L^1(P_X)$ for all $(t, z) \in \mathbf{S}_0$, $\|\ell - \Upsilon(\kappa_j)\|_{\bar{Q},1} = o(1)$, (63), and Lemma A.5 imply there are functions $\{f_0(t, z, X)\}_{(t,z) \in \mathbf{S}_0}$ satisfying $f_0(t, z, \cdot) \in L^1(P_X)$ and

$$E_{\bar{Q}}\left[\left|\ell(T^*, X) - \sum_{(t,z) \in \mathbf{S}_0} 1\{T^*(z) = t\}f_0(t, z, X)\right|\right] = 0. \tag{64}$$

Finally, set $\kappa(t, z, X) \equiv f_0(t, z, X)/P(Z = z|X)$ and note $\kappa \in L^1(P_{TZX})$ because $P(Z = z|X) \geq \varepsilon > 0$ and $f_0(t, z, \cdot) \in L^1(P_X)$. By result (63) and direct calculation, $\|\ell - \Upsilon(\kappa)\|_{\bar{Q},1} = 0$, yielding that identification of λ_Q implies the existence of the desired κ and hence establishing part (i) of the corollary. Part (ii) is immediate from part (i), Lemma A.2, $\mu \ll \bar{Q}$ by assumption, and $\bar{Q} \ll \mu$ since $\bar{Q} \in \Theta_0$. *Q.E.D.*

PROOF OF THEOREM 4.3: We first show that if ℓ belongs to the τ-closure of $\mathcal{R}_t$, then λ_{Q_0} is identified. To this end, note $\mathcal{B}$ and $\mathcal{R}_t$ are identified because Θ_0 and $\Upsilon : L^1(P) \to L^1(\bar{Q})$ are identified. Hence, Theorem 2.14 in Aliprantis and Border (2006) and ℓ being in the τ-closure of $\mathcal{R}_t$ imply there is an identified net $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subseteq L^1(P_{tZX})$ satisfying

$$\lim_\alpha \langle \Upsilon(f_\alpha), s\rangle_Q = \langle \ell, s\rangle_Q \quad \text{for all } s \in \mathcal{S}_Q, Q \in \Theta_0. \tag{65}$$

Next, let $\Pi_Q(s)(T^*, X) \equiv E_Q[s(Y^*, T^*, X)|T^*, X]$ for any $s \in L^1(Q)$, and note that $\lambda_Q = \langle \ell, \Pi_Q(\rho)\rangle_Q$. Therefore, for any $Q \in \Theta_0$, $\Pi_Q(\rho) \in \mathcal{S}_Q$ by Assumption 4.4(i), result (65), and $f_\alpha(T, Z, X) = 1\{T = t\}g_\alpha(Z, X)$ for some g_α by definition of $\mathcal{R}_t$ imply

$$\begin{aligned}
\lambda_Q &= \lim_\alpha E_Q\big[E_Q\big[\rho\big(Y^*(t), X\big)|T^*, X\big]E_{P_{Z|X}}\big[g_\alpha(Z, X)1\big\{T^*(Z) = t\big\}\big]\big] \\
&= \lim_\alpha E_P\big[\rho(Y, X)g_\alpha(Z, X)1\{T = t\}\big],
\end{aligned} \tag{66}$$

where the final equality follows from Corollary A.1, the law of iterated expectations, and $Q \in \Theta_0$. Since (66) holds for any $Q \in \Theta_0$, it follows that λ_{Q_0} is identified.

We next show λ_{Q_0} being identified implies ℓ belongs to the τ-closure of $\mathcal{R}_t$. To this end, we first note that Theorem 4.1 and λ_{Q_0} being identified imply $\ell\rho$ belongs to the τ-closure of $\mathcal{R}$. By Theorem 2.14 in Aliprantis and Border (2006), then

$$\lim_{\alpha}\big\langle \Upsilon(v_\alpha), s\big\rangle_Q = \langle \ell\rho, s\rangle_Q \quad \text{for all } s \in \mathcal{S}_Q, Q \in \Theta_0 \tag{67}$$

for some net $\{v_\alpha\}_{\alpha\in\mathcal{A}} \subseteq L^1(P)$. Moreover, $Y^*(t) \perp\!\!\!\perp T^*|X$ under $\bar{Q}^{\text{io}}$, $\bar{Q}_{Y^*(t)X} = \bar{Q}^{\text{io}}_{Y^*(t)X}$ and Assumptions 4.4(ii)(iii) imply $\bar{Q}^{\text{io}}(E_{\bar{Q}^{\text{io}}}[\rho(Y^*(t), X)\phi_{\bar{Q},\rho}(Y^*(t), X)|T^*, X] = 1) = 1$. Therefore, for any $s \in \mathcal{S}_{\bar{Q}}$, the law of iterated expectations and $\bar{Q}_{Y^*(t)X} = \bar{Q}^{\text{io}}_{Y^*(t)X}$ imply

$$\langle \ell, s\rangle_{\bar{Q}} = \big\langle \ell\rho, \phi_{\bar{Q},\rho}\Pi_{\bar{Q}}(s)\big\rangle_{\bar{Q}^{\text{io}}} = \left\langle \ell\rho, \phi_{\bar{Q},\rho}\Pi_{\bar{Q}}(s)\frac{d\bar{Q}^{\text{io}}_{Y^*T^*X}}{d\bar{Q}_{Y^*T^*X}}\right\rangle_{\bar{Q}}, \tag{68}$$

where the final equality used Assumption 4.4(ii). Moreover, since 4.4(i)(iii) imply that $\phi_{\bar{Q},\rho}\Pi_{\bar{Q}}(s)(d\bar{Q}^{\text{io}}_{Y^*T^*X}/d\bar{Q}_{Y^*T^*X}) \in \mathcal{S}_{\bar{Q}}$ for any $s \in \mathcal{S}_{\bar{Q}}$, we have

$$\langle \ell, s\rangle_{\bar{Q}} = \lim_{\alpha}\left\langle \Upsilon(v_\alpha), \phi_{\bar{Q},\rho}\Pi_{\bar{Q}}(s)\frac{d\bar{Q}^{\text{io}}_{Y^*T^*X}}{d\bar{Q}_{Y^*T^*X}}\right\rangle_{\bar{Q}} = \lim_{\alpha}\big\langle \Upsilon(v_\alpha), \phi_{\bar{Q},\rho}\Pi_{\bar{Q}}(s)\big\rangle_{\bar{Q}^{\text{io}}} \tag{69}$$

due to (67) and (68). Next, set $g_\alpha(Z, X) \equiv E_{\bar{Q}_{Y^*(t)|X}}[v_\alpha(Y^*(t), t, Z, X)\phi_{\bar{Q},\rho}(Y^*(t), X)]$ and $f_\alpha(T, Z, X) \equiv 1\{T = t\}g_\alpha(Z, X)$. Also note that $\bar{Q} \in \Theta_0$, Jensen's inequality, $\phi_{\bar{Q},\rho} \in L^\infty(\bar{Q})$ by Assumption 4.4(iii), and the definition of $\bar{Q}^{\text{io}}$ imply that

$$\begin{aligned}
E_P\big[\big|f_\alpha(T, Z, X)\big|\big] &\lesssim E_{\bar{Q}_{T^*ZX}}\big[1\big\{T^*(Z) = t\big\}E_{\bar{Q}_{Y^*(t)|X}}\big[\big|v_\alpha\big(Y^*(t), t, Z, X\big)\big|\big] \\
&= E_{\bar{Q}^{\text{io}}}\big[1\big\{T^*(Z) = t\big\}\big|v_\alpha\big(Y^*(t), t, Z, X\big)\big|\big] \\
&\lesssim E_P\big[\big|1\{T = t\}v_\alpha(Y, T, Z, X)\big|\big],
\end{aligned} \tag{70}$$

where the final inequality follows from $\bar{Q} \in \Theta_0$ and $d\bar{Q}^{\text{io}}/d\bar{Q} \in L^\infty(\bar{Q})$ by Assumption 4.4(ii). Since $v_\alpha \in L^1(P)$, (70) implies $\Upsilon(f_\alpha) \in \mathcal{R}_t$. Also, the law of iterated expectations, $\bar{Q} \in \Theta_0$, Corollary A.1, $\bar{Q}^{\text{io}}_{T^*ZX} = \bar{Q}_{T^*ZX}$, and the definition of f_α imply for any $s \in \mathcal{S}_{\bar{Q}}$

$$\begin{aligned}
&\big\langle \Upsilon(f_\alpha), s\big\rangle_{\bar{Q}} \\
&\quad = E_{\bar{Q}^{\text{io}}}\big[v_\alpha\big(Y^*(t), t, Z, X\big)1\big\{T^*(Z) = t\big\}\phi_{\bar{Q},\rho}\big(Y^*(t), X\big)E_{\bar{Q}}\big[s\big(Y^*, T^*, X\big)|T^*, X\big]\big] \\
&\quad = \sum_{\tilde{t}\in\mathbf{T}} E_{\bar{Q}^{\text{io}}}\big[v_\alpha\big(Y^*(\tilde{t}), \tilde{t}, Z, X\big)1\big\{T^*(Z) = \tilde{t}\big\}\phi_{\bar{Q},\rho}\big(Y^*(t), X\big)E_{\bar{Q}}\big[s\big(Y^*, T^*, X\big)|T^*, X\big]\big] \\
&\quad = \big\langle \Upsilon(v_\alpha), \phi_{\bar{Q},\rho}\Pi_{\bar{Q}}(s)\big\rangle_{\bar{Q}^{\text{io}}},
\end{aligned} \tag{71}$$

where the second equality follows from $(Y^*(\tilde{t}), T^*, Z)$ being independent of $Y^*(t)$ conditionally on X under $\bar{Q}^{\text{io}}$ whenever $\tilde{t} \neq t$, $E_{\bar{Q}^{\text{io}}}[\phi_{\bar{Q},\rho}(Y^*(t), X)|X] = 0$ and $\bar{Q}^{\text{io}}_{Y^*(t)X} = \bar{Q}_{Y^*(t)X}$; and the final equality holds by Lemma A.1 and $\bar{Q}^{\text{io}}_{ZX} = \bar{Q}_{ZX} = P_{ZX}$. To conclude, note that for any $Q \in \Theta_0$ and $s \in \mathcal{S}_Q$, $Q \ll \bar{Q}$, $\Pi_Q(s)\, dQ_{T^*X}/d\bar{Q}_{T^*X} \in \mathcal{S}_{\bar{Q}}$ by Assumptions 4.4(i)(iv), and results (69) and (71) yield that

$$\langle \ell, s\rangle_Q = \left\langle \ell, \Pi_Q(s)\frac{dQ_{T^*X}}{d\bar{Q}_{T^*X}}\right\rangle_{\bar{Q}} = \lim_\alpha \left\langle \Upsilon(f_\alpha), \Pi_Q(s)\frac{dQ_{T^*X}}{d\bar{Q}_{T^*X}}\right\rangle_{\bar{Q}} = \lim_\alpha \langle \Upsilon(f_\alpha), s\rangle_Q.$$

Hence, since $\Upsilon(f_\alpha) \in \mathcal{R}_t$, we conclude that ℓ belongs to the τ-closure of $\mathcal{R}_t$. *Q.E.D.*

PROOF OF COROLLARY 4.4: The proof is similar to that of Corollary 4.3 and we therefore omit details. In particular, the same arguments used in Corollary 4.3 here imply that the conditions of Theorem 4.3 are satisfied. Therefore, if (i) holds, then Theorem 4.3 and Lemma A.4 yield that there is $\{\kappa_j\} \in L^1(P_{ZXt})$ satisfying $\|\ell - \Upsilon(\kappa_j)\|_{\bar{Q},1} = o(1)$.

Next, let $\mathbf{Z}_0 \equiv \{z \in \mathbf{Z} : P(T = t, Z = z) > 0\}$, note $\kappa_j(T, Z, X) = 1\{T = t\}g_j(Z, X)$ for some g_j, and use the same arguments in Corollary 4.3 to show $g_j(z, \cdot) \in L^1(P_X)$ for any $z \in \mathbf{Z}_0$. Setting $\tilde{g}_j(z, X) \equiv g_j(z, X)P(Z = z|X)$, we obtain by definition of Υ,

$$\|\ell - \Upsilon(\kappa_j)\|_{\bar{Q},1} = E_{\bar{Q}}\left[\left|\ell(T^*, X) - \sum_{z\in\mathbf{Z}_0} 1\{T^*(z) = t\}\tilde{g}_j(z, X)\right|\right]. \tag{72}$$

Therefore, $\|\ell - \Upsilon(\kappa_j)\|_{\bar{Q},1} = o(1)$, result (72), and Lemma A.5 allow us to conclude

$$E_{\bar{Q}}\left[\left|\ell(T^*, X) - \sum_{z\in\mathbf{Z}_0} 1\{T^*(z) = t\}f_0(z, X)\right|\right] = 0 \tag{73}$$

for some $\{f_0(z, X)\}_{z\in\mathbf{Z}_0}$ satisfying $f_0(z, \cdot) \in L^1(P_X)$ for all $z \in \mathbf{Z}_0$. Hence, setting $\kappa(T, Z, X) \equiv 1\{T = t\}\sum_{z\in\mathbf{Z}_0} 1\{Z = z\}f_0(Z, X)/P(Z = z|X)$, we obtain from $P(Z = z|X) \geq \varepsilon > 0$ that $\kappa \in L^1(P_{tZX})$ and from (73) that $\bar{Q}(\Upsilon(\kappa) = \ell) = 1$. Since $\bar{Q}(\Upsilon(\kappa) = \ell) = 1$ implies $\bar{Q}(\Upsilon(f\kappa) = f\ell) = 1$ for any bounded f, Lemma 4.2 yields that (i) implies (iii). Because (iii) implies (ii) and (ii) implies (i), the corollary follows. *Q.E.D.*

COROLLARY A.1: *If Assumption 2.1 holds, $Q \in \Theta_0$, and $f \in L^1(P)$, then it follows*

$$E_Q[f(Y, T, Z, X)|Y^*, T^*, X] = \sum_{t\in\mathbf{T}} E_{P_{Z|X}}[f(Y^*(t), t, Z, X)1\{T^*(Z) = t\}].$$

PROOF: It follows from Lemma A.1, $Q_{Z|X} = P_{Z|X}$ due to $Q \in \Theta_0$, and $Y = Y^*(T)$ and $T = T^*(Z)$ implying that $f(Y, T, Z, X) = \sum_{t\in\mathbf{T}} f(Y^*(t), t, Z, X)1\{T^*(Z) = t\}$. *Q.E.D.*

LEMMA A.1: *If $(Y^*, T^*) \perp\!\!\!\perp Z|X$ under Q, then $E_Q[f(Y^*, T^*, Z, X)|Y^*, T^*, X] = E_{Q_{Z|X}}[f(Y^*, T^*, Z, X)]$ for any $f \in L^1(Q)$.*

PROOF: Let $\mathcal{G}$ be the σ-field generated by $(\mathcal{G}_{Y^*} \times \mathcal{G}_{T^*} \times \mathcal{G}_Z \times \mathcal{G}_X)$ where $\mathcal{G}_V$ denotes the σ-field on which the marginal distribution Q_V is defined. We further define the class

$$\mathcal{A} \equiv \{A \in \mathcal{G} : E_Q[1\{(Y^*, T^*, Z, X) \in A\}|Y^*, T^*, X] = E_{Q_{Z|X}}[1\{(Y^*, T^*, Z, X) \in A\}]\}$$

and note that $\mathbf{Y}^* \times \mathbf{T}^* \times \mathbf{Z} \times \mathbf{X} \in \mathcal{A}$. Also, observe that if $A_1, A_2 \in \mathcal{A}$ and $A_1 \subseteq A_2$, then $A_2 \setminus A_1 \in \mathcal{A}$. Next, let $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$ be a sequence of pairwise disjoint sets and note

$$\begin{aligned}
&E_Q\left[1\left\{(Y^*, T^*, Z, X) \in \bigcup_{i=1}^\infty A_i\right\} | Y^*, T^*, X\right] \\
&\quad = \lim_{n\to\infty} E_{Q_{Z|X}}\left[\sum_{i=1}^n 1\{(Y^*, T^*, Z, X) \in A_i\}\right] \\
&\quad = E_{Q_{Z|X}}\left[1\left\{(Y^*, T^*, Z, X) \in \bigcup_{i=1}^\infty A_i\right\}\right], \qquad (74)
\end{aligned}$$

by Theorem 10.1.5(4) in Bogachev (2007), $\{A_i\}_{i=1}^\infty$ being disjoint, and $A_i \in \mathcal{A}$ for all i. In particular, (74) implies $\bigcup_{i=1}^\infty A_i \in \mathcal{A}$, and we can therefore conclude that $\mathcal{A}$ is a λ-system.

Let $A_{Y^*} \in \mathcal{G}_{Y^*}$, $A_{T^*} \in \mathcal{G}_{T^*}$, $A_Z \in \mathcal{G}_Z$, and $A_X \in \mathcal{G}_X$ be arbitrary, and then observe

$$\begin{aligned}
&E_Q[1\{(Y^*, T^*, Z, X) \in A_{Y^*} \times A_{T^*} \times A_Z \times A_X\} | Y^*, T^*, X] \\
&\quad = E_{Q_{Z|X}}[1\{(Y^*, T^*, Z, X) \in A_{Y^*} \times A_{T^*} \times A_Z \times A_X\}] \qquad (75)
\end{aligned}$$

due to $(Y^*, T^*) \perp\!\!\!\perp Z|X$ under Q. Result (75) implies $A_{Y^*} \times A_{T^*} \times A_Z \times A_X \in \mathcal{A}$ and therefore that $(\mathcal{G}_{Y^*} \times \mathcal{G}_{T^*} \times \mathcal{G}_Z \times \mathcal{G}_X) \subseteq \mathcal{A}$. Since $(\mathcal{G}_Z \times \mathcal{G}_X \times \mathcal{G}_{T^*} \times \mathcal{G}_{Y^*})$ is a π-system, the $\pi - \lambda$ theorem (see, e.g., Theorem 2.38 in Pollard (2002)) implies $\mathcal{A} = \mathcal{G}$.

To conclude, let $f \in L^1(Q)$ and $\{f_n\}_{n=1}^\infty$ be a sequence of simple functions satisfying $|f_n| \leq |f|$ and $f_n(Y^*, T^*, Z, X) \to f(Y^*, T^*, Z, X)$ on a set with Q-probability one. By Proposition 10.1.7 in Bogachev (2007), we can then conclude that

$$\begin{aligned}
E_Q[f(Y^*, T^*, Z, X)|Y^*, T^*, X] &= \lim_{n\to\infty} E_Q[f_n(Y^*, T^*, Z, X)|Y^*, T^*, X] \\
&= \lim_{n\to\infty} E_{Q_{Z|X}}[f_n(Y^*, T^*, Z, X)] \\
&= E_{Q_{Z|X}}[f(Y^*, T^*, Z, X)],
\end{aligned}$$

where the second equality holds due to f_n being a simple function and $\mathcal{A} = \mathcal{G}$. *Q.E.D.*

LEMMA A.2: *Let Assumptions 2.1 and 2.2 hold, and suppose that $\mu(\pi(Z, X) > \delta) = 1$ for some $\delta > 0$. If $\nu \in L^1(P)$, then it follows $\kappa = \nu/\pi$ satisfies $\kappa \in L^1(P)$ and*

$$E_{\mu_{Z|X}}[\nu(Y^*(t), t, Z, X)1\{T^*(Z) = t\}] = E_{P_{Z|X}}[\kappa(Y^*(t), t, Z, X)1\{T^*(Z) = t\}].$$

PROOF: Since $\bar{Q} \ll \mu$ and $\bar{Q}_{ZX} = P_{ZX}$ because $\bar{Q} \in \Theta_0$, it follows that $\mu(\pi(Z, X) > \delta) = 1$ implies $P(\pi(Z, X) > \delta) = 1$. Hence, $1/\pi \in L^\infty(P)$, which together with $\nu \in L^1(P)$ yields $\kappa = \nu/\pi \in L^1(P)$. The lemma therefore follows by the definition of κ and π. *Q.E.D.*

LEMMA A.3: *Let Assumptions 2.1, 2.2, and 4.1(ii) hold, and suppose that a finite collection $\{s_j, Q_j\}_{j=1}^J$ with $s_j \in \mathcal{S}_{Q_j}$ and $Q_j \in \Theta_0$ for all $1 \leq j \leq J$ satisfies*

$$P\left(\sum_{j=1}^J E_{Q_j}[s_j(Y^*, T^*, X)|Y, T, Z, X] = 0\right) = 1. \qquad (76)$$

Then there is a $Q^{\mathrm{a}} \in \Theta_0$ *and a* $\eta > 0$ *such that the measure* $\tilde{Q}$ *given by* $\tilde{Q}(A) \equiv Q^{\mathrm{a}}(A) + \sum_{j=1}^{J} \eta E_{Q_j}[s_j(Y^*, T^*, X)1\{(Y^*, T^*, Z, X) \in A\}]$ *belongs to* Θ_0.

PROOF: By Assumption 4.1(ii), there is a $Q^{\mathrm{i}} \in \Theta_0$ such that $dQ^{\mathrm{i}}/d\mu$ belongs to the interior of $\mathcal{Q}$ in $\mathbf{Q}$. Set $Q^{\mathrm{a}} \equiv \lambda Q^{\mathrm{i}} + (1-\lambda)\sum_{j=1}^{J} Q_j/J$ and note $dQ^{\mathrm{a}}/d\mu$ belongs to the interior of $\mathcal{Q}$ in $\mathbf{Q}$ provided $\lambda \in (0,1)$ is chosen sufficiently large, and $Q^{\mathrm{a}} \in \Theta_0$ due to Θ_0 being convex by Lemma 4.1. Since $\|s_j\|_{Q_j,\infty} < \infty$ for all $1 \leq j \leq J$ because $s_j \in \mathcal{S}_{Q_j} \subseteq L^{\infty}(Q_j)$, the definition of Q^{a} implies $\tilde{Q}(A) \geq \sum_{j=1}^{J} Q_j(A)((1-\lambda)/J - \eta\|s_j\|_{Q_j,\infty})$. Hence, since $\lambda < 1$, $\tilde{Q}$ is a positive measure provided we set $\eta > 0$ sufficiently small. Further observe

$$\tilde{Q}(\mathbf{Y}^* \times \mathbf{T}^* \times \mathbf{Z} \times \mathbf{X}) = Q^{\mathrm{a}}(\mathbf{Y}^* \times \mathbf{T}^* \times \mathbf{Z} \times \mathbf{X}) + \eta \sum_{j=1}^{J} E_{Q_j}[s_j(Y^*, T^*, X)] = 1 \tag{77}$$

by (76), the law of iterated expectations, and $Q_j \in \Theta_0$. It follows that $\tilde{Q}$ is a probability measure. Moreover, since $Q_j \ll \mu$ due to $Q_j \in \Theta_0$, we have $\tilde{Q} \ll \mu$ and $d\tilde{Q}/d\mu = dQ^{\mathrm{a}}/d\mu + \eta \sum_j s_j\, dQ_j/d\mu$. Therefore, $dQ^{\mathrm{a}}/d\mu$ belonging to the interior of $\mathcal{Q}$ in $\mathbf{Q}$ and $s_j\, dQ_j/d\mu \in \mathbf{Q}$ by definition of $\mathcal{S}_{Q_j}$ imply that $d\tilde{Q}/d\mu \in \mathcal{Q}$ for $\eta > 0$ small.

Next, note that for any $t \in \mathbf{T}$ and (measurable) set V, condition (76) and $Q_j \in \Theta_0$ imply

$$\begin{aligned}
&\sum_{j=1}^{J} E_{Q_j}\big[s_j(Y^*, T^*, X)1\{T^*(Z) = t, (Y^*(t), Z, X) \in V)\}\big] \\
&\quad = \sum_{j=1}^{J} E_{Q_j}\big[E_{Q_j}[s_j(Y^*, T^*, X)|Y, T, Z, X]1\{T = t, (Y, Z, X) \in V\}\big] = 0.
\end{aligned} \tag{78}$$

Result (78), $Q^{\mathrm{a}} \in \Theta_0$, and the definition of $\tilde{Q}$ imply $\tilde{Q}$ induces P. To conclude the proof, it remains to show $(Y^*, T^*) \perp\!\!\!\perp Z|X$ under $\tilde{Q}$. Let $f \in L^1(P_{ZX})$ and g be a bounded function of (Y^*, T^*, X). Since $(Y^*, T^*) \perp\!\!\!\perp Z|X$ under Q^{a} and Q_j, due to $Q^{\mathrm{a}}, Q_j \in \Theta_0$, we have

$$\begin{aligned}
&E_{\tilde{Q}}\big[g(Y^*, T^*, X)f(Z, X)\big] \\
&\quad = E_{Q^{\mathrm{a}}}\big[g(Y^*, T^*, X)E_{Q^{\mathrm{a}}}[f(Z, X)|X]\big] \\
&\qquad + \eta \sum_{j=1}^{J} E_{Q_j}\big[g(Y^*, T^*, X)s_j(Y^*, T^*, X)E_{Q_j}[f(Z, X)|X]\big].
\end{aligned} \tag{79}$$

However, since we showed $\tilde{Q}$ is observationally equivalent to Q_0 and Q^{a} and Q_j are observationally equivalent to Q_0 (due to $Q^{\mathrm{a}}, Q_j \in \Theta_0$) it also follows that $E_{Q^{\mathrm{a}}}[f(Z, X)|X] = E_{Q_j}[f(Z, X)|X] = E_{\tilde{Q}}[f(Z, X)|X]$. Combining this observation with (79) then yields

$$E_{\tilde{Q}}\big[g(Y^*, T^*, X)f(Z, X)\big] = E_{\tilde{Q}}\big[g(Y^*, T^*, X)E_{\tilde{Q}}[f(Z, X)|X]\big]. \tag{80}$$

Since (80) holds for any bounded g, it follows $E_{\tilde{Q}}[f(Z, X)|Y^*, T^*, X] = E_{\tilde{Q}}[f(Z, X)|X]$; see, for example, Definition 10.1.1 in Bogachev (2007). Thus, since $f \in L^1(P_{ZX})$ was also arbitrary, we conclude $(Y^*, T^*) \perp\!\!\!\perp Z|X$ under $\tilde{Q}$ and, therefore, that $\tilde{Q} \in \Theta_0$. *Q.E.D.*

LEMMA A.4: *Let Assumptions 2.1, 2.2 hold,* $\ell \in \bigcap_{Q\in\Theta_0} L^1(Q)$, $\mathcal{S}_{\bar{Q}} = L^\infty(\bar{Q}_{Y^*T^*X})$, *and* $\mathcal{C} \subseteq \mathcal{R}$ *be convex. If* ℓ *belongs to the* τ*-closure of* $\mathcal{C}$, *then there is a sequence* $\{f_n\}_{n=1}^\infty \subseteq L^1(\bar{Q}_{Y^*T^*X})$ *such that* $\lim_{n\to\infty} \|\ell - f_n\|_{\bar{Q},1} = 0$.

PROOF: By Theorem 2.14 in Aliprantis and Border (2006) and ℓ belonging to the τ-closure of $\mathcal{C}$, there is a net $\{c_\alpha\}_{\alpha\in\mathcal{A}} \subseteq \mathcal{C}$ such that $\lim_\alpha \langle s, c_\alpha\rangle_{\bar{Q}} = \langle s, \ell\rangle_{\bar{Q}}$ for all $s \in \mathcal{S}_{\bar{Q}} = L^\infty(\bar{Q}_{Y^*T^*X})$. A second application of Theorem 2.14 in Aliprantis and Border (2006) implies $\ell \in L^1(\bar{Q}_{Y^*T^*X})$ belongs to the closure of $\mathcal{C} \subseteq L^1(\bar{Q}_{Y^*T^*X})$ under the weak topology generated by $\mathcal{S}_{\bar{Q}} = L^\infty(\bar{Q}_{Y^*T^*X})$. However, since $L^\infty(\bar{Q}_{Y^*T^*X})$ is the norm dual of $L^1(\bar{Q}_{Y^*T^*X})$ and $\mathcal{C} \subseteq L^1(\bar{Q}_{Y^*T^*X})$ is convex, it follows that ℓ belongs to the closure of $\mathcal{C}$ under $\|\cdot\|_{\bar{Q},1}$ (see, e.g., Theorem 3.12 in Rudin (1991)). Therefore, by Theorem 2.40(1) in Aliprantis and Border (2006), $\|\ell - f_n\|_{\bar{Q},1} = o(1)$ for some$\{f_n\}_{n=1}^\infty \subseteq C$. *Q.E.D.*

LEMMA A.5: *Let Assumptions 2.1 and 2.2 hold,* $\mathbf{T}^*$ *be finite,* $\{C_i\}_{i=1}^r$ *be a finite collection of subsets of* $\mathbf{T}^*$, *and define* $A : \bigotimes_{i=1}^r L^1(P_X) \to L^1(\bar{Q}_{T^*X})$ *by setting* $A(f)(T^*, X) = \sum_{i=1}^r 1\{T^* \in C_i\} f_i(X)$ *for any* $f = \{f_i\}_{i=1}^r \in \bigotimes_{i=1}^r L^1(P_X)$. *If* $\bar{Q}(T^* = t^*|X) \geq \varepsilon > 0$ *a.s. for any* $t^* \in \mathbf{T}^*$ *satisfying* $\bar{Q}(T^* = t^*) > 0$, *then the range of* A *is closed.*

PROOF: Let $\mathbf{T}_0^* \equiv \{t^* \in \mathbf{T}^* : \bar{Q}(T^* = t^*) > 0\}$ and enumerate $\mathbf{T}_0^*$ by $\mathbf{T}_0^* = \{t_1^*, \ldots, t_{d^*}^*\}$. Interpret any $\{f_i\}_{i=1}^r \in \bigotimes_{i=1}^r L^1(P_X)$ as a column vector $f(X) \equiv (f_1(X), \ldots, f_r(X))'$ and define a $d^* \times r$ matrix Ω whose jth row is given by $\omega_j \equiv (1\{t_j^* \in C_1\}, \ldots, 1\{t_j^* \in C_r\})$. Then note that $\bar{Q}(T^* = t^*|X) \geq \varepsilon > 0$ for all $t^* \in \mathbf{T}_0^*$ and $\bar{Q}_X = P_X$ due to $\bar{Q} \in \Theta_0$ imply

$$E_{\bar{Q}}\big[\big|A(f)(T^*, X)\big|\big] \geq \varepsilon E_{P_X}\big[\big\|\Omega f(X)\big\|_1\big] \geq \varepsilon\eta E_{P_X}\big[\big\|\Omega^\dagger\Omega f(X)\big\|_1\big], \tag{81}$$

where the final inequality holds for some $\eta > 0$ and $\Omega^\dagger$ the Moore–Penrose pseudoinverse of Ω because $\Omega\Omega^\dagger\Omega = \Omega$ and $\Omega : \text{range}\{\Omega^\dagger\} \to \mathbf{R}^{d^*}$ is invertible. Next, suppose $\|A(f_n) - \ell\|_{\bar{Q},1} = o(1)$ for some $\{f_n\}_{n=1}^\infty \in \bigotimes_{i=1}^r L^1(P_X)$ and $\ell \in L^1(\bar{Q}_{T^*X})$. Result (81) implies $\{\Omega^\dagger\Omega f_n\}_{n=1}^\infty$ is Cauchy in $\bigotimes_{i=1}^r L^1(P_X)$ and, therefore, by completeness of $\bigotimes_{i=1}^r L^1(P_X)$, there is $\tilde{f} \in \bigotimes_{i=1}^r L^1(P_X)$ satisfying $E_{P_X}[\|\Omega^\dagger\Omega f_n(X) - \tilde{f}(X)\|_1] = o(1)$. The same manipulations as in result (81) and $\Omega = \Omega\Omega^\dagger\Omega$ then imply that

$$\begin{aligned}
&E_{\bar{Q}}\big[\big|\ell(T^*, X) - A(\tilde{f})(T^*, X)\big|\big] \\
&\quad\leq \lim_{n\to\infty} E_{\bar{Q}}\big[\big|A(f_n - \tilde{f})(T^*, X)\big|\big] \\
&\quad\leq \lim_{n\to\infty} E_{P_X}\big[\big\|\Omega f_n(X) - \Omega\tilde{f}(X)\big\|_1\big] \leq \lim_{n\to\infty} \|\Omega\|_{o,1} E_{P_X}\big[\big\|\Omega^\dagger\Omega f_n(X) - \tilde{f}(X)\big\|_1\big] = 0,
\end{aligned}$$

where $\|\cdot\|_{o,1}$ denotes the operator norm of $\Omega : (\mathbf{R}^r, \|\cdot\|_1) \to (\mathbf{R}^{d^*}, \|\cdot\|_1)$ and the equality follows from $E_{P_X}[\|\Omega^\dagger\Omega f_n(X) - \tilde{f}(X)\|_1] = o(1)$. Hence, the range of A is closed. *Q.E.D.*

APPENDIX B: PROOFS FOR SECTION 5

PROOF OF THEOREM 5.1: The first equality in (38) follows from the definitions of $\hat{\lambda}$ and $\hat{\psi}_k$ and applying Lemma B.1 with W_i satisfying $P(W_i = 1) = 1$. Since $\|\psi\|_\infty \lesssim B$ by Assumption 5.2(i) and $\sigma^2 = \text{Var}_P\{\psi(Y, T, Z, X)\}$ by definition, Theorem 1.1 in Zhai (2018) yields

$$\frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n}\big(\psi(Y_i,T_i,Z_i,X_i)-E_P\big[\psi(Y,T,Z,X)\big]\big)=\mathbb{Z}+O_P\bigg(\frac{B\log(n)}{\sigma\sqrt{n}}\bigg) \tag{82}$$

for $\mathbb{Z}$ a standard normal random variable possibly depending on n. The second equality in (38) then follows from (82), Lemma B.2, and $B\log(n)=o(\sigma\sqrt{n})$ by Assumption 5.2(i). *Q.E.D.*

PROOF OF THEOREM 5.2: First, note that Lemma B.1, $(\hat{\lambda}-\lambda_{Q_0})=O_P(\sigma/\sqrt{n})$ by Theorem 5.1, and $\sum_i W_i/n=o_P(1)$ due to $E[W]=0$ together allow us to conclude that

$$\frac{\sqrt{n}}{\sigma}(\hat{\lambda}^*-\hat{\lambda})=\frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n}W_i\psi(Y_i,T_i,Z_i,X_i)+o_P(1). \tag{83}$$

Next, note that $\sigma^2\equiv\mathrm{Var}_P\{\psi(Y,T,Z,X)\}$ by definition, $\|\psi\|_\infty\lesssim B$ and $B\log(n)=o(\sigma\sqrt{n})$ by Assumption 5.2(i), and Bernstein's inequality yield that for n sufficiently large

$$P\Bigg(\bigg|\frac{1}{n\sigma}\sum_{i=1}^{n}\big(\psi(Y_i,T_i,Z_i,X_i)-E_P\big[\psi(Y,T,Z,X)\big]\big)\bigg|>\frac{\log(n)}{\sqrt{n}}\Bigg) \le 2\exp\bigg\{-\frac{\log^2(n)}{4}\bigg\}. \tag{84}$$

In particular, note that result (84) together with Lemma B.2 allow us to conclude that

$$\bigg|\frac{1}{n}\sum_{i=1}^{n}\big\{\psi^2(Y_i,T_i,Z_i,X_i)-\big(\psi(Y_i,T_i,Z_i,X_i)-E_P\big[\psi(Y,T,Z,X)\big]\big)^2\big\}\bigg| = o_P\bigg(\frac{\sigma^2\log(n)}{n}\bigg). \tag{85}$$

Moreover, Markov's inequality, $\|\psi\|_\infty\lesssim B$, and $\sigma^2\equiv\mathrm{Var}_P\{\psi(Y,T,Z,X)\}$ further imply

$$P\Bigg(\bigg|\frac{1}{n}\sum_{i=1}^{n}\frac{\big(\psi(Y_i,T_i,Z_i,X_i)-E_P\big[\psi(Y,T,Z,X)\big]\big)^2}{\sigma^2}-1\bigg|>\epsilon\Bigg)\lesssim\frac{B^2\sigma^2}{n\sigma^4\epsilon^2} \tag{86}$$

for any $\epsilon>0$. Therefore, setting $\hat{\sigma}^2\equiv\sum_i\psi^2(Y_i,T_i,Z_i,X_i)/n$, we can conclude from results (85) and (86) together with $B\log(n)=o(\sigma\sqrt{n})$ by Assumption 5.2(i) that

$$\frac{\hat{\sigma}^2}{\sigma^2}=\frac{1}{n}\sum_{i=1}^{n}\frac{\big(\psi(Y_i,T_i,Z_i,X_i)-E_P\big[\psi(Y,T,Z,X)\big]\big)^2}{\sigma^2}+o_P(1)=1+o_P(1). \tag{87}$$

To conclude, note that $\{W_i\}_{i=1}^n$ being i.i.d. standard normal random variables and $\{W_i\}_{i=1}^n$ being independent of $\{Y_i,T_i,Z_i,X_i\}_{i=1}^n$ imply $\mathbb{Z}^*\equiv(n\hat{\sigma}^2)^{-1/2}\sum_{i=1}^n W_i\psi(Y_i,T_i,Z_i,X_i)$ satisfies $\mathbb{Z}^*\sim N(0,1)$ conditionally on $\{Y_i,T_i,Z_i,X_i\}_{i=1}^n$, and hence, $\mathbb{Z}^*$ is independent of $\{Y_i,T_i,Z_i,X_i\}_{i=1}^n$. The theorem therefore follows from (83) and (87). *Q.E.D.*

LEMMA B.1: *Let Assumptions 5.1(i)(iii) and 5.2(i)(ii)(v) hold, $\{W_i\}_{i=1}^n$ be an i.i.d. sequence independent of $\{Y_i, T_i, Z_i, X_i\}_{i=1}^n$ satisfying $E[W^2] < \infty$, ψ and $\hat{\psi}_k$ be as defined in (37) and (39), respectively, and $\sigma^2 \equiv \mathrm{Var}_P\{\psi(Y, T, Z, X)\}$. Then it follows that*

$$\frac{1}{n}\sum_{k=1}^{K}\sum_{i\in I_k} W_i\{\hat{\psi}_k(Y_i, T_i, Z_i, X_i) + \hat{\lambda}\} = \frac{1}{n}\sum_{i=1}^{n} W_i\{\psi(Y_i, T_i, Z_i, X_i) + \lambda_{Q_0}\} + o_P\left(\frac{\sigma}{\sqrt{n}}\right).$$

PROOF: Let $\hat{\Delta}_k^\gamma \equiv (\hat{\gamma}_k - \gamma)$, $\hat{\Delta}_k^\beta \equiv (\hat{\beta}_k - \beta)$, and note for any k we have the decomposition

$$\begin{aligned}
&\frac{1}{n}\sum_{i\in I_k} W_i\{(\hat{\psi}_k(Y_i, T_i, Z_i, X_i) + \hat{\lambda}) - (\psi(Y_i, T_i, Z_i, X_i) + \lambda_{Q_0})\} \\
&\quad = \frac{1}{n}\sum_{i\in I_k} W_i(\nu_j(Y_i, T_i, Z_i, X_i) - b(Z_i, X_i)'\beta)b(Z_i, X_i)'\hat{\Delta}_k^\gamma \\
&\qquad + \frac{1}{n}\sum_{i\in I_k} W_i\{E_{\mu_{Z|X}}[b(Z, X_i)'] - b(Z_i, X_i)'\gamma b(Z_i, X_i)'\}\hat{\Delta}_k^\beta \\
&\qquad - \frac{1}{n}\sum_{i\in I_k} W_i(\hat{\Delta}_k^\gamma)'b(Z_i, X_i)b(Z_i, X_i)'\hat{\Delta}_k^\beta.
\end{aligned} \tag{88}$$

Next, note that $E_P[(\nu_j(Y, T, Z, X) - b(Z, X)'\beta)b(Z, X)] = 0$ as a result of the first-order conditions for β. Hence, since $\hat{\gamma}_k$ is computed using the observations in I_k^c, we obtain

$$E\left[\frac{1}{n}\sum_{i\in I_k} W_i(\nu_j(Y_i, T_i, Z_i, X_i) - b(Z_i, X_i)'\beta)b(Z_i, X_i)'\hat{\Delta}_k^\gamma|\hat{\gamma}_k\right] = 0.$$

Therefore, the observations within I_k being i.i.d., $|I_k| \asymp n$ by Assumption 5.2(v), $\|b'\beta\|_\infty \vee \|\nu_j\|_\infty \equiv B < \infty$ by Assumption 5.2(i), $E[W^2] < \infty$, and the definition of r^γ yield

$$\mathrm{Var}\left\{\frac{1}{n}\sum_{i\in I_k} W_i(\nu_j(Y_i, T_i, Z_i, X_i) - b(Z_i, X_i)'\beta)b(Z_i, X_i)'\hat{\Delta}_k^\gamma|\hat{\gamma}_k\right\} \lesssim \frac{B^2 r_\gamma^2}{n}. \tag{89}$$

Moreover, by identical arguments but relying on the first-order condition for γ yields that

$$\begin{aligned}
&\mathrm{Var}\left\{\frac{1}{n}\sum_{i\in I_k} W_i\{E_{\mu_{Z|X}}[b(Z, X_i)'] - b(Z_i, X_i)'\gamma b(Z_i, X_i)'\}\hat{\Delta}_k^\beta|\hat{\beta}_k\right\} \\
&\quad \lesssim \frac{1}{n}\{E_P[(E_{\mu_{Z|X}} b(Z, X)'\hat{\Delta}_k^\beta])^2] + E_P[(b(Z, X)'\gamma)^2(b(Z, X)'\hat{\Delta}_k^\beta)^2] \lesssim \frac{r_\beta^2}{n},
\end{aligned} \tag{90}$$

where in the final inequality we employed Jensen's inequality, Assumptions 5.1(iii) and 5.2(i), and $d\mu_{Z|X}/dP_{Z|X} = 1/\pi$. Finally, note that the Cauchy–Schwarz inequality yields

$$\left|\frac{1}{n}\sum_{i\in I_k} W_i(\hat{\Delta}_k^\gamma)'b(Z_i, X_i)b(Z_i, X_i)'\hat{\Delta}_k^\beta\right|$$

$$\leq \left\{\frac{1}{n}\sum_{i\in I_k} W_i^2\big(b(Z_i, X_i)'\hat{\Delta}_k^{\gamma}\big)^2\right\}^{1/2} \times \left\{\frac{1}{n}\sum_{i\in I_k}\big(b(Z_i, X_i)'\hat{\Delta}_k^{\beta}\big)^2\right\}^{1/2}. \tag{91}$$

The lemma follows from results (88), (89), (90), and (91), $E[W_i^2] < \infty$, $\{W_i\}_{i=1}^n$ being independent of $\{Y_i, T_i, Z_i, X_i\}_{i=1}^n$, Markov's inequality and Assumption 5.2(ii). *Q.E.D.*

LEMMA B.2: *If Assumptions* 5.1(*ii*), 5.2(*iii*)(*iv*) *hold*, *and* ψ *is as defined in* (37), *then it follows that* $|E_P[\psi(Y, T, Z, X)]| = o(\sigma/\sqrt{n})$.

PROOF: First, note that the first-order conditions for γ imply $E_{P_X}[E_{\mu_{Z|X}}[b(Z, X)'\beta]] - E_P[(b(Z, X)'\gamma)(b(Z, X)'\beta)] = 0$. Together with the definition of ψ, the law of iterated expectations, and $\kappa_j = \nu_j/\pi$, this first-order condition therefore implies that

$$\begin{aligned}&\big|E_P\big[\psi(Y, T, Z, X)\big] - \big(E_P\big[\kappa_j(Y, T, Z, X)\big] - \lambda_{Q_0}\big)\big|\\ &\quad = \left|E_P\left[\left(b(Z, X)'\gamma - \frac{1}{\pi(Z, X)}\right)\big(E_P\big[\nu_j(Y, T, Z, X)|Z, X\big] - b(Z, X)'\beta\big)\right]\right|. \end{aligned} \tag{92}$$

Therefore, the claim of the lemma follows from result (92), the Cauchy–Schwarz inequality, and Assumptions 5.2(iii)(iv). *Q.E.D.*

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All authors assume responsibility for all aspects of the paper.